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Analytic Semigroups Generated by Non-variational Elliptic Systems of Second Order under Dirichlet Boundary Conditions

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1. INTRODUCTION

Let Ω be an open bounded domain in $\mathbb{R}^n$, with boundary of class C^2, and let us consider the second-order differential operator

$$Eu = -\sum_{i,j=1}^{n} A_{ij} D_{ij} u + \sum_{j=1}^{n} B_j D_j u + Cu \tag{1.1}$$

under the Dirichlet boundary condition

$$u = 0, \qquad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here u is a complex vector with N components and the coefficients A_{ij}, B_j, and C are $N \times N$ complex valued matrices.

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Let us suppose that the operator E is elliptic, in the sense that there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{n} \xi_i \xi_j \operatorname{Re}(A_{ij}(x)\, \eta | \eta)_N \geqslant \nu |\xi|_n^2 |\eta|_N^2$$

for each $x \in \Omega$, each $\xi \in \mathbb{R}^n$, and each $\eta \in \mathbb{C}^N$.

As for the regularity of the coefficients of E, the matrices A_{ij} are assumed to be continuous in $\bar{\Omega}$, whereas B_j and C may only be bounded and measurable in Ω, or even be less regular.

In this paper we are interested in the problem of the generation of analytic semigroups by the elliptic operator $-E$ in various Banach space topologies, under Dirichlet boundary conditions.

The problem, as it is set in the L^2 topology, has been considered by several authors who have usually concentrated their attention on the case $N = 1$. In Section 3 of this paper we have considered the same problem, but our attention has been rather directed to the case of $N \geqslant 1$. We have done this in order to achieve a more complete analysis and also because it is necessary to the development of further arguments.

Our interest was attracted first by the semigroup generation in the Hölder topology.

In that period results for that case were not completely satisfactory, as Stewart also pointed out in [18].

In fact, von Wahl [20, 21] remarked that no estimate of the form

$$|\lambda|(|\lambda - \omega)\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant C\|f\|_{C^{0,\alpha}(\bar{\Omega})} \tag{1.3}$$

holds, in general, for the solutions of the elliptic system

$$(\lambda + E)\, u = f, \qquad \operatorname{Re} \lambda > \omega, \tag{1.4}$$

under Dirichlet boundary conditions.

Following this, Campanato [10, 12] analyzed elliptic scalar equations of divergence form. He proved an estimate of the type (1.3) in two cases: either under Neumann boundary conditions, or, under condition (1.2), assuming that f vanishes on $\partial\Omega$ as well.

His method comes from the theory of regularity for solutions of elliptic and parabolic systems in the spaces $\mathscr{L}^{2,\mu}(\Omega)$. This theory, which makes no use of the techniques of [4], was developed by Campanato in several works (see [11] for a systematic exposition) and is frequently used to obtain Hölder estimates, due to the fact that $\mathscr{L}^{2,\mu}(\Omega) = C^{0,(\mu-n)/2}(\bar{\Omega})$ for $\mu \in]n, n+2]$.

We then decided to seek generation results in the $\mathscr{L}^{2,\mu}$ topology in the case of second-order elliptic operators of the form (1.1), not only if

$\mu \in]n, n+2[$, but also for $\mu \in]0, n[$, when $\mathscr{L}^{2,\mu}(\Omega)$ is isomorphic to the Morrey space $L^{2,\mu}(\Omega)$.

In both case we obtain estimates of the following kind

$$(|\lambda| - \omega)\|u\|_{\mathscr{L}^{2,\mu}(\Omega)} + (|\lambda| - \omega)^{1/2} \sum_j \|D_j u\|_{\mathscr{L}^{2,\mu}(\Omega)} + \sum_{|\alpha| = 2} \|D^\alpha u\|_{\mathscr{L}^{2,\mu}(\Omega)} \leqslant C\|f\|_{\mathscr{L}^{2,\mu}(\Omega)} \tag{1.5}$$

for the solutions of system (1.4) under condition (1.2).

Inequality (1.5) not only yields an immediate characterization of the operator's domain—at least when $\mu \in [0, n[$—but is also useful in several applications of the theory of analytic semigroups to parabolic systems (see, for instance, [1, 16–18]).

The generation theorem in Morrey spaces is contained in Section 5, together with the estimate (1.5) for $\mu \in]0, n[$. Here the choice of the homogeneous boundary condition (1.2) has been done just in order to simplify the exposition. Nonhomogeneous Dirichlet boundary conditions may be disposed of similarly.

From the results of Section 5 we deduce the generation in the uniform topology for $N \geqslant 1$. This theorem was proved by Stewart [17, 18] under general boundary conditions for the case $N = 1$. This deduction may be found in Section 6 (Theorem 6.1). There, by a suitable interpolation procedure, we obtain once more, for systems of the form (1.1), the L^p estimate due to Agmon [3].

Finally, in Section 7 we consider the case of Hölder spaces and we prove inequality (1.5) when $\mu \in]n, n+2[$. As has already been remarked, condition (1.2) is now essential and has to be satisfied by vector f as well. Moreover, since estimate (1.5) involves second-order derivatives, the coefficients A_{ij}, B_j, and C of the operator have to be Hölder continuous, together with the boundary of Ω.

However, as we will show in the future, it is possible to obtain a weaker estimate of the form (1.3) without these stronger conditions.

2. Notations and Preliminary Results

In this section we recall some basic facts concerning elliptic systems, which will be frequently used in the sequel.

If $N \geqslant 1$ is an integer, we denote by $(|)_N$ and $| \, |_N$, respectively, the scalar product and the norm in $\mathbb{C}^N$, using the same symbols for the scalar product and the norm in $\mathbb{R}^N$ when the vectors are real. We shall also omit the subscript N when there is no danger of confusion. Moreover, if M is a $N \times N$ complex matrix, we denote by $|M|$ the norm of M as an element of $\mathbb{C}^{N^2}$.

If $x_0 \in \mathbb{R}^n$ and $\sigma > 0$, let us set

$$B(x_0, \sigma) = \{x \in \mathbb{R}^n : |x_0 - x| < \sigma\}$$
$$B(0, \sigma) = B(\sigma).$$

If we write, for $x \in \mathbb{R}^n$, $x = (x', x_n)$, let us also set

$$B^+(\sigma) = \{x \in B(\sigma) : x_n > 0\}$$
$$B^-(\sigma) = \{x \in B(\sigma) : x_n < 0\}$$
$$\Gamma(\sigma) = \{x \in B(\sigma) : x_n = 0\}.$$

For most of this paper we assume that *Ω is an open bounded domain in $\mathbb{R}^n$ with boundary of class C^2*, which means that there exists a finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding set $\{\Phi_j\}$ of one-to-one transformations taking U_j onto $B(1)$ such that:

for each j, Φ_j and Φ_j^{-1} have continuous second-order derivatives, respectively, on $\overline{U_j}$ and $\overline{B(1)}$; (2.1)

for each j, $\Phi_j(U_j \cap \Omega) = B^+(1)$, $\Phi_j(U_j \cap \partial\Omega) = \Gamma(1)$. (2.2)

Of course, most of the results of the present section hold for much less regular domains.

Let d_Ω denote the diameter of Ω.

We now recall a few properties of some function spaces, a systematic exposition of which may be found in [5, 6, 11] (see also [13]).

Let A be a measurable subset of $\mathbb{R}^n$, with positive measure.

If $u: A \to \mathbb{C}^N$ is integrable over A, let us set

$$u_A = (\operatorname{meas} A)^{-1} \int_A u(x)\, dx.$$

If $0 \leqslant \mu \leqslant n+2$, we denote by $\mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N)$ the Banach space of the vectors $u: \Omega \to \mathbb{C}^N$, such that $u \in L^2(\Omega, \mathbb{C}^N)$ and

$$[u]^2_{\mathscr{L}^{2,\mu}(\Omega)} = \sup_{x \in \Omega, 0 < \sigma \leqslant d_\Omega} \sigma^{-\mu} \int_{\Omega \cap B(x,\sigma)} |u(y) - u_{\Omega \cap B(x,\sigma)}|^2 dy < +\infty$$

(see also [8]). The norm of $\mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N)$ is the following:

$$\|u\|_{\mathscr{L}^{2,\mu}(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{\mathscr{L}^{2,\mu}(\Omega)}.$$

We say that $u \in \mathscr{L}_0^{2,\mu}(\Omega, \mathbb{C}^N)$ if $u \in \mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N)$ and

$$\sup_{x \in \partial\Omega, 0 < \sigma \leqslant d_\Omega} \sigma^{-\mu} \int_{\Omega \cap B(x,\sigma)} |u(y)|^2 dy < +\infty.$$

If $0 \leqslant \mu \leqslant n$, we denote by $L^{2,\mu}(\Omega, \mathbb{C}^N)$ the Morrey space of the vectors $u \in L^2(\Omega, \mathbb{C}^N)$, such that

$$\|u\|^2_{\mathscr{L}^{2,\mu}(\Omega)} = \sup_{x \in \Omega, 0 < \sigma \leqslant d_\Omega} \sigma^{-\mu} \int_{\Omega \cap B(x,\sigma)} |u(y)|^2 dy + \infty$$

(see, for instance, [11]).

If $0 < \alpha \leqslant 1$, then $u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$ means that $u: \bar{\Omega} \to \mathbb{C}^N$ and

$$[u]_{C^{0,\alpha}(\bar{\Omega})} = \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|_N}{|x - y|_n^\alpha} < +\infty.$$

$C^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$ is a Banach space with the following norm:

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\alpha}(\bar{\Omega})}.$$

Finally, $u \in C_0^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$ means that $u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$ and $u = 0$ on $\partial\Omega$.

The following result is due to Campanato [5].

THEOREM 2.1. *Assume that Ω has the cone property. If $0 \leqslant \mu < n$, then $\mathscr{L}_0^{2,\mu}(\Omega, \mathbb{C}^N) = \mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N) = L^{2,\mu}(\Omega, \mathbb{C}^N)$ and*

$$[u]_{\mathscr{L}^{2,\mu}(\Omega)} \leqslant \|u\|_{L^{2,\mu}(\Omega)} \leqslant C(\Omega, \mu) \|u\|_{\mathscr{L}^{2,\mu}(\Omega)}.$$

If $n < \mu \leqslant n + 2$, then $\mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N) = C^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$, $\mathscr{L}_0^{2,\mu}(\Omega, \mathbb{C}^N) = C_0^{0,\alpha}(\bar{\Omega}, \mathbb{C}^N)$ with $\alpha = (\mu - n)/2$ and

$$C(n)[u]_{\mathscr{L}^{2,\mu}(\Omega)} \leqslant [u]_{C^{0,\alpha}(\bar{\Omega})} \leqslant C(\Omega, n)[u]_{\mathscr{L}^{2,\mu}(\Omega)}.$$

Remark 2.2. It is easy to show that, for each $0 \leqslant \mu < n$,

$$L^{2,n}(\Omega, \mathbb{C}^N) = L^\infty(\Omega, \mathbb{C}^N) \subsetneq \mathscr{L}^{2,n}(\Omega, \mathbb{C}^N)$$
$$L^{2,n}(\Omega, \mathbb{C}^N) \subset L^{2,\mu}(\Omega, \mathbb{C}^N)$$

and

$$\|u\|_{L^{2,\mu}(\Omega)} \leqslant c d_\Omega^{(n-\mu)/2} \|u\|_{L^\infty(\Omega)}.$$

It is also easy to show that, for each $0 \leqslant \mu < n$,

$$L^{2,\mu}(\Omega, \mathbb{C}^N) \not\subset L^p(\Omega, \mathbb{C}^N)$$

for any $p > 2$.

Let us set $D_i = \partial/\partial x_i$, $D_{ij} = D_i D_j$, $i, j = 1,..., n$, and, if α is the multi-index $(\alpha_1,..., \alpha_n)$, let us also set

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

Let $1 \leqslant p < +\infty$ be a real number and $m \geqslant 0$ be an integer. We denote by $H^{m,p}(\Omega, \mathbb{C}^N)$ and $H_0^{m,p}(\Omega, \mathbb{C}^N)$ the usual Sobolev spaces, which are the closure, respectively, of $C^\infty(\bar{\Omega}, \mathbb{C}^N)$ and of $C_0^\infty(\Omega, \mathbb{C}^N)$ in the topology determined by the norm

$$\|u\|_{m,p,\Omega} = \left[\sum_{j=0}^{m} |u|_{j,p,\Omega}^2 \right]^{1/2}$$

where we have set

$$|u|_{j,p,\Omega} = \left[\int_\Omega \left(\sum_{|\alpha|=j} |D^\alpha u|^2 \right)^{p/2} dx \right]^{1/p}$$

recalling that $|\alpha| = \alpha_1 + \cdots + \alpha_n$. In particular, $H^{0,p}(\Omega, \mathbb{C}^N) = L^p(\Omega, \mathbb{C}^N)$ and

$$|u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \|u\|_{L^p(\Omega)}.$$

Remark 2.3. It is well known that, if Ω has the segment property, then $H^{m,p}(\Omega, \mathbb{C}^N)$ can be characterized as the space of the vectors $u: \Omega \to \mathbb{C}^N$, whose distributional derivatives belong to $L^p(\Omega, \mathbb{C}^N)$ up to the order m (see, for instance, [2, Sect. 3.18]). We also denote by $H^{m,\infty}(\Omega, \mathbb{C}^N)$ the space of the vectors $u: \Omega \to \mathbb{C}^N$ such that

$$|u|_{j,p,\Omega} = \sup_{x \in \Omega} \left[\sum_{|\alpha|=j} |D^\alpha u(x)|^2 \right]^{1/2} < +\infty$$

for each $j = 0, 1, \ldots, m$.

If $p = 2$, we simply write $H^m(\Omega, \mathbb{C}^N)$, $H_0^m(\Omega, \mathbb{C}^N)$ and, for the norm and the seminorms, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{j,\Omega}$.

$H^m(\Omega, \mathbb{C}^N)$, $m \geqslant 0$, is a Hilbert space with the inner product

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha| \leqslant m} (D^\alpha u \mid D^\alpha v)_N \, dx.$$

If $0 \leqslant \mu \leqslant n+2$, we denote by $H^m_{(\mu)}(\Omega, \mathbb{C}^N)$ the space of the vectors $u \in H^m(\Omega, \mathbb{C}^N)$ such that $D^\alpha u \in \mathscr{L}^{2,\mu}(\Omega, \mathbb{C}^N)$, for each α with $|\alpha| = m$.

Remark 2.4. Using Poincaré's inequality, one immediately proves that, if Ω is convex and $u \in H^1_{(\mu)}(\Omega, \mathbb{C}^N)$, $0 \leqslant \mu < n$, then $u \in \mathscr{L}^{2,\mu+2}(\Omega, \mathbb{C}^N)$ and

$$[u]_{\mathscr{L}^{2,\mu+2}(\Omega)} \leqslant c(\Omega, n) \sum_{i=1}^{n} \|D_i u\|_{L^{2,\mu}(\Omega)}$$

(see also [11, Lemma 3.III, Chap. 1]).

From [8, Lemma 2.IV] and from the fact that $|\cdot|_{2,\Omega}$ is an equivalent norm in $H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$, one can easily deduce the following interpolation inequalities.

LEMMA 2.5. *Assume that Ω has the cone property. If $u \in H^2_{(\mu)} \cap H_0^1(\Omega, \mathbb{C}^N)$ with $0 \leqslant \mu < n$, then*

$$\sum_{i=1}^{n} \|D_i u\|_{L^{2,\mu}(\Omega)} \leqslant c(\Omega, n)\, \|u\|^{1/2}_{L^{2,\mu}(\Omega)} \cdot \left(\sum_{|\beta|=2} \|D^\beta u\|_{L^{2,\mu}(\Omega)} \right)^{1/2}. \tag{2.3}$$

If $u \in H^2_{(\mu)} \cap H_0^1(\Omega, \mathbb{C}^N)$ with $n < \mu \leqslant n+2$, then

$$\sum_{i=1}^{n} \|D_i u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant c(\Omega, n)\, \|u\|^{1/2}_{C^{0,\alpha}(\bar{\Omega})} \left(\sum_{|\beta|=2} \|D^\beta u\|_{C^{0,\alpha}(\bar{\Omega})} \right)^{1/2} \tag{2.4}$$

where $\alpha = (\mu - n)/2$.

Let A_{ij}, B_i $(i, j = 1,..., n)$, and C be $N \times N$ complex valued matrices, such that

A_{ij} are continuous in $\bar{\Omega}$; (2.5)

B_i and C are bounded and measurable in Ω. (2.6)

Let us assume that the following *ellipticity condition* holds: *there exists $\nu > 0$ such that for each $x \in \Omega$, for each $\xi \in \mathbb{R}^n$, and each $\eta \in \mathbb{C}^N$*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \operatorname{Re}(A_{ij}(x)\, \eta | \eta)_N \geqslant |\xi|_n^2 |\eta|_N^2. \tag{2.7}$$

Consider the elliptic operator in divergence form

$$\mathscr{E} u = -\sum_{i,j=1}^{n} D_i(A_{ij} D_j u) + \sum_{i=1}^{n} B_i D_i u + Cu$$

and the corresponding sesquilinear form on $H^1(\Omega, \mathbb{C}^N) \times H^1(\Omega, \mathbb{C}^N)$

$$a_\Omega(u, \phi) = \int_\Omega \left\{ \sum_{i,j=1}^{n} A_{ij} D_j u \,|\, D_i \phi)_N + \sum_{i=1}^{n} (B_i D_i u \,|\, \phi)_N + (Cu \,|\, \phi)_N \right\} dx.$$

Let us set

$$L = \sup_{x \in \Omega} \left[\sum_{ij=1}^{n} |A_{ij}(x)|^2 \right]^{1/2}, \qquad \lambda_0 = \frac{1}{\nu} \sup_{x \in \Omega} \sum_{i=1}^{n} |B_i(x)|^2 + \sup_{x \in \Omega} |C(x)|. \tag{2.8}$$

Then, it is well known that, *for each* $u \in H_0^1(\Omega, \mathbb{C}^N)$,

$$\operatorname{Re} a_\Omega(u, u) \geqslant (\nu/2)|u|_{1,\Omega}^2 - (\lambda + \Lambda)|u|_{0,\Omega}^2 \tag{2.9}$$

where Λ is a positive constant depending on A_{ij} and Ω.

Remark 2.6. From Gårding inequality (2.9) we conclude that, if $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant \lambda_0 + \Lambda$, then for each $f \in L^2(\Omega, \mathbb{C}^N)$ there exists a unique solution of the Dirichlet problem

$$\begin{aligned} &u \in H_0^1(\Omega, \mathbb{C}^N) \\ &\lambda(u, \phi)_{0,\Omega} + a_\Omega(u, \phi) = (f, \phi)_{0,\Omega}, \qquad \forall \phi \in H_0^1(\Omega, \mathbb{C}^N). \end{aligned} \tag{2.10}$$

By a standard argument one also obtains

$$(|\lambda| - \lambda_0)|u|_{0,\Omega} \leqslant c|f|_{0,\Omega} \tag{2.11}$$

where c depends only the maximum between ν^{-1} and L.

Using inequality (2.9) and the translation technique, the following version of the classical differentiability theorem can be proved:

THEOREM 2.7. *Assume that $\partial\Omega$ is of class C^2 and let $u \in H_0^1(\Omega, \mathbb{C}^N)$ be a solution of the Dirichlet problem* (2.10) *with $f \in L^2(\Omega, \mathbb{C}^N)$ and $\lambda \in \mathbb{C}$.*

If conditions (2.6) *and* (2.7) *hold and if the coefficients A_{ij} are of class C^1 in $\bar{\Omega}$, then $u \in H^2(\Omega, \mathbb{C}^N)$. Moreover, if* $\operatorname{Re} \lambda \geqslant \lambda_0 + \Lambda$, *then*

$$|u|_{2,\Omega} \leqslant c_1|u|_{0,\Omega} + c_2|f|_{0,\Omega}. \tag{2.12}$$

Here c_1 does not depend on λ or u and $c_2 = c_2(\Omega, L \vee \nu^{-1})$.

For the reader's convenience we give a proof of (2.12) in the Appendix.

Remark 2.8. Let us suppose, in particular, that the coefficients A_{ij} are *constant* and so that $\Lambda = 0$. Choosing in (2.10) $\phi = u$, one obtains that, if $\operatorname{Re} \lambda \geqslant \lambda_0$, then

$$(\nu/2)|u|_{1,\Omega}^2 \leqslant |f|_{0,\Omega}|u|_{0,\Omega}.$$

Therefore, recalling Poincaré's inequality in $H_0^1(\Omega, \mathbb{C}^N)$

$$|u|_{0,\Omega} \leqslant c(\Omega)|u|_{1,\Omega},$$

one concludes that

$$|u|_{0,\Omega} \leqslant \frac{2c^2(\Omega)}{\nu}|f|_{0,\Omega}.$$

From this inequality and from (2.12) it follows that

$$|u|_{2,\Omega} \leqslant c|f|_{0,\Omega}$$

with c independent of λ and u. It is then easy to show that the constant c does not vary under homothetical transformations.

Consider now the non-variational operator associated with the matrices A_{ij}, B_i, and C:

$$Eu = -\sum_{ij=1}^{n} A_{ij} D_{ij} u + \sum_{i=1}^{n} B_i D_i u + Cu.$$

Using the previous results and the contraction mapping theorem, the following two lemmas can be deduced, to guarantee existence on "small" balls and half-balls:

LEMMA 2.9. *Let us suppose that the operator E satisfies the hypotheses* (2.5), (2.6), *and* (2.7) *in $\Omega = B(r)$. Then there exists $\sigma_0 \in]0, r]$ such that, for each $\lambda \in \mathbb{C}$ with* Re $\lambda \geqslant \lambda_0$ *and each $\sigma \in]0, \sigma_0]$, the Dirichlet problem*

$$u \in H^2 \cap H_0^1(B(\sigma), C^N)$$
$$(\lambda + E)\, u = f \in L^2(B(\sigma), \mathbb{C}^N)$$

has a unique solution u and

$$(|\lambda| - \lambda_0)|u|_{0,B(\sigma)} + |u|_{2,B(\sigma)} \leqslant c|f|_{0,B(\sigma)} \tag{2.13}$$

where c is a constant independent of λ, σ, and u.

LEMMA 2.10. *Let us suppose that the operator E satisfies the hypotheses* (2.5), (2.6), *and* (2.7) *in $\Omega = B^+(1)$. Then there exists $\sigma_0 \in]0, 1]$ such that, for each $\lambda \in \mathbb{C}$ with* Re $\lambda \geqslant \lambda_0$ *and each $\sigma \in]0, \sigma_0]$, the problem*

$$u \in H^2(B^+(\sigma), \mathbb{C}^N), \qquad u = 0 \text{ on } \Gamma(\sigma)$$
$$(\lambda + E)\, u = f \in L^2(B^+(\sigma), \mathbb{C}^N)$$

has a solution u such that

$$(|\lambda| - \lambda)|u|_{0,B^+(\sigma)} + |u|_{2,B^+(\sigma)} \leqslant c|f|_{0,B^+(\sigma)} \tag{2.14}$$

and

$$|u|_{1,B^+(\sigma)} \leqslant c\sigma|f|_{0,B^+(\sigma)} \tag{2.15}$$

where the constant c does not depend on λ, σ, and u.

Inequalities that bound the derivatives of a certain order in terms of the lower-order derivatives are usually known as Caccioppoli-type inequalities.

The Caccioppoli-type inequalities we need in the present work are contained in the two lemmas below. Their proof has much in common with that of [12, Lemmas 1.III and 1.IV] and so we will only sketch it for the reader's convenience.

LEMMA 2.11. *Let $u \in H^2(B(R), \mathbb{C}^N)$ be a solution of the system*

$$\lambda u - \sum_{i,j} A_{ij}^0 D_{ij} u = \zeta \in \mathbb{C}^N \qquad \text{in} \quad B(R) \tag{2.16}$$

where $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda \geqslant 0$ and A_{ij}^0 are constant elliptic matrices. Then, for each $\rho \in]0, R[$,

$$|u|_{2,B(\rho)} \leqslant \frac{C}{R-\rho} |u|_{1,B(R)} \tag{2.17}$$

and

$$|u|_{1,B(\rho)} \leqslant \frac{C}{R-\rho} \left(\frac{R}{\rho}\right)^{n/2} |u - u_{B(R)}|_{0,B(R)} \tag{2.18}$$

where the constant c does not depend on λ, R, ρ, and u.

Proof. Let us show (2.17) first. We remark that, as $u \in C^\infty(B(R), \mathbb{C}^N)$, we can differentiate system (2.16), thus obtaining, for each $s = 1,..., n$,

$$\lambda(D_s u) - \sum_{i,j} A_{ij}^0 D_{ij}(D_s u) = 0 \qquad \textit{in} \quad B(R)$$

or, equivalently,

$$\lambda \int_{B(R)} (D_s u \mid \phi)_N dx + \int_{B(R)} \sum_{i,j} (A_{ij}^0 D_j(D_s u) \mid D_i \phi)_N dx = 0,$$
$$\forall \phi \in C_0^\infty(B(R), \mathbb{C}^N).$$

Let now $\theta \in C_0^\infty(B(R), \mathbb{R})$ be a usual cutoff function:

$$0 \leqslant \theta \leqslant 1,\ \theta \equiv 1 \text{ on } B(\rho), \qquad |\operatorname{grad} \theta| \leqslant C/(R-\rho). \tag{2.19}$$

Choosing $\phi = \theta^2 D_s u$ we get

$$\lambda \int_{B(R)} \theta^2 |D_s u|_N^2 dx + \int_{B(R)} \sum_{ij} (A_{ij}^0 D_j[\theta D_s u] | D_i[\theta D_s u])\, dx$$

$$= \int_{B(R)} \Big\{ \sum_{ij} (A_{ij}^0 D_s u | D_i[\theta D_s u])\, D_j\theta$$

$$- \sum_{ij} (A_{ij}^0 D_j[\theta D_s u] | D_s u)\, D_i\theta$$

$$+ \sum_{ij} D_i\theta D_j\theta (A_{ij}^0 D_s u | D_s u) \Big\}\, dx.$$

Then, recalling (2.9) we have, because of (2.19),

$$\operatorname{Re} \lambda |\theta D_s u|_{0,B(R)}^2 + \frac{\nu}{2} |\theta D_s u|_{1,B(R)}^2$$

$$\leqslant \varepsilon |\theta D_s u|_{1,B(R)}^2 + \frac{C(\varepsilon, L)}{(R-\rho)^2} |D_s u|_{0,B(R)}^2$$

for each $\varepsilon > 0$. Choosing ε sufficiently small, (2.17) is proved.

To show (2.18), consider the weak form of (2.16):

$$\lambda \int_{B(R)} (u | \phi)_N dx + \int_{B(R)} \sum_{ij} (A_{ij}^0 D_j u | D_i \phi)_N dx$$

$$= \int_{B(R)} (\zeta | \phi)_N dx, \quad \forall \phi \in H_0^1(B(R), \mathbb{C}^N).$$

Taking

$$\phi = \theta^2 (u - u_{\theta, B(R)})$$

where

$$u_{\theta, B(R)} = \left(\int_{B(R)} \theta^2(x)\, dx \right)^{-1} \int_{B(R)} \theta^2(x)\, u(x)\, dx,$$

so that

$$\int_{B(R)} \theta^2 (u | u - u_{\theta, B(R)})_N dx = \int_{B(R)} \theta^2 |u - u_{\theta, B(R)}|_N^2 dx$$

and

$$\int_{B(R)} \theta^2 (\zeta | u - u_{\theta, B(R)})_N dx = 0,$$

one easily obtains (2.18), just as in [12, Lemma 1.III]. ∎

LEMMA 2.12. *Let $u \in H^2(B^+(R), \mathbb{C}^N)$ be such that $u = 0$ on $\Gamma(R)$ and*

$$\lambda u - \sum_{ij=1}^{n} A_{ij}^0 D_{ij} u + \sum_{i=1}^{n} B_i^0 D_i u = 0 \qquad \text{in } B^+(R) \tag{2.20}$$

where A_{ij}^0, B_i^0 are constant matrices and A_{ij}^0 are elliptic, with ellipticity constant ν. If

$$\operatorname{Re} \lambda \geqslant \nu^{-1} \sum_i |B_i^0|^2$$

then, for each $\rho \in]0, R[$,

$$|u|_{1, B^+(\rho)} \leqslant \frac{C}{R-\rho} |u|_{0, B^+(R)} \tag{2.21}$$

and

$$\sum_{s=1}^{n-1} |D_s u|_{1, B^+}(\rho) \leqslant \frac{C}{R-\rho} |u|_{1, B^+(R)} \tag{2.22}$$

where the constant c does not depend on λ, R, ρ, and u.

The proof of Lemma 2.12 is similar to that of the previous one. One considers the weak form of system (2.20):

$$\lambda \int_{B^+(R)} (u \mid \phi)\, dx + \int_{B^+(R)} \left\{ \sum_{ij} (A_{ij}^0 D_j u \mid D_i \phi) \right.$$
$$\left. + \sum_i (B_i^0 D_i u \mid \phi) \right\} dx = 0 \tag{2.23}$$

for each $\phi \in H_0^1(B^+(R), \mathbb{C}^N)$. The choice of $\phi = \theta^2 u$, with θ satisfying condition (2.19), gives inequality (2.21).

Inequality (2.22) can be deduced from (2.21), noting that $D_s u$, for $1 \leqslant s \leqslant n-1$, still solves system (2.23) and vanishes on $\Gamma(R)$.

For the following propositions see [11, Lemmas 1.I and 1.III]. They are a basic technical tool:

LEMMA 2.13. *Let ϕ and Φ be non-negative functions defined in $]0, d]$, let Φ be non-decreasing, and let A, α, and β be positive constants with $\beta < \alpha$. Assume that for each $t \in]0, 1]$ and each $\sigma \in]0, d]$*

$$\phi(t\sigma) \leqslant A t^{\alpha} \phi(\sigma) + \sigma^{\beta} \Phi(\sigma).$$

Then for each $\varepsilon \in]0, \alpha-\beta]$, *for each* $t \in]0,1]$, *and each* $\sigma \in]0, d]$

$$\phi(t\sigma) \leqslant At^{\alpha-\varepsilon}\phi(\sigma) + K(A)(t\sigma)^{\beta}\Phi(\sigma)$$

where

$$K(\xi) = \frac{(1+\xi)^{2\alpha/\varepsilon}}{(1+\xi)^{(\alpha-\beta)/\varepsilon} - \xi}.$$

LEMMA 2.14. *Let* ϕ *and* θ *be non-negative functions defined in* $]0, d]$ *and let* A *and* α *be positive constants. If*

$$\lim_{\sigma \to 0} \theta(\sigma) = 0$$

and

$$\phi(t\sigma) \leqslant (At^{\alpha} + \theta(\sigma))\, \phi(\sigma)$$

for each $t \in]0, 1[$ *and* $\sigma \in]0, d]$, *then for each* $\varepsilon > 0$ *there exists* $d_\varepsilon \in]0, d]$ *such that*

$$\phi(t\sigma) \leqslant (1+A)\, t^{\alpha-\varepsilon}\phi(\sigma)$$

for each $t \in]0, 1[$ *and* $\sigma \in]0, d_\varepsilon]$.

We conclude this section by recalling some well-known L^p regularity results for solutions of the elliptic system

$$-\sum_{ij} A_{ij} D_{ij} u = f. \tag{2.24}$$

Let us initially assume that A_{ij} are *constant* matrices. It is well known that, if $f \in L^p(B(r), \mathbb{C}^N)$, $p \geqslant 2$, then system (2.24) has a unique solution in the space $H^{2,p} \cap H_0^{1,p}(B(r), \mathbb{C}^N)$ and

$$|u|_{2,p,B(r)} \leqslant c|f|_{0,p,B(r)} \tag{2.25}$$

where u denotes such a solution and c does not depend on r.

This result has been proved in many different ways. In [7], for example, it is obtained as a consequence of regularization in $\mathscr{L}^{2,n}$ and of Stampacchia's interpolation theorem (see [11, Chap. II, No. 7] for the case of systems; see also [19, Chap. III, No. 6]).

A standard consequence of inequality (2.25), via the contraction mapping theorem, is the following:

LEMMA 2.15. *Let matrices* A_{ij} *be elliptic and continuous in* $B(1)$ *and let* $f \in L^p(B(1), \mathbb{C}^N)$, $2 \leqslant p$.

Then there exists $\sigma_0 \in]0, 1]$ *such that system* (2.24) *has a unique solution in the space* $H^{2,p} \cap H_0^{1,p}(B(\sigma), \mathbb{C}^N)$ *for each* $\sigma \in]0, \sigma_0]$. *Moreover, denoting by u such a solution,*

$$|u|_{2,p,B(\sigma)} \leqslant c|f|_{0,p,B(\sigma)} \tag{2.26}$$

where c is independent of σ.

Remark 2.16. Lemma 2.16 has been stated for balls for the sake of simplicity. As a matter of fact, (2.26) holds in much more general solutions.

For instance, let A be an open bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary ($\partial\Omega$ of class C^3 would certainly do). Assume that $B^+(1/2) \subset A \subset B^+(1)$ and denote by $A(\sigma)$ the set onto which A is mapped by the homothetical transformation $x \to \sigma x$. Then the conclusions of Lemma 2.15 remain true if $B(1)$ is replaced by A and $B(\sigma)$ by $A(\sigma)$.

3. The Estimate in the Space $L^2(\Omega, \mathbb{C}^N)$

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary of class C^2.

In this section we prove a theorem concerning the generation of analytic semigroup in $L^2(\Omega, \mathbb{C}^N)$. This result is well known, mainly when $N = 1$ (see, for instance, [3]), and could be deduced from "local" propositions. We prefer, however, to give a straightforward proof, based only on the differentiability theorem stated in the previous section (Theorem 2.7).

Let us consider the second-order differential operator

$$Eu = -\sum_{ij=1}^{n} A_{ij} D_{ij} u + \sum_{j=1}^{n} B_j D_j u + Cu$$

where u is a vector of $\mathbb{C}^N$, defined in Ω, and where A_{ij}, B_j, and C are $N \times N$ complex valued matrices. Assume that the matrices A_{ij} are continuous in $\bar{\Omega}$ and elliptic in the sense of condition (2.7), whereas B_j and C, such as in No. 2, may be supposed only bounded and measurable in Ω.

We are interested in the Dirichlet problem

$$\begin{aligned} &u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N) \\ &(\lambda + E)\, u = f \in L^2(\Omega, \mathbb{C}^N) \end{aligned} \tag{3.1}$$

where λ is a complex number.

Consider $N \times N$ complex matrices $\tilde{A}_{ij}$, which are of class C^1 in $\bar{\Omega}$ and which satisfy the ellipticity condition (2.7).

Problem (3.1) is, of course, equivalent to the problem

$$u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$$
$$u - \sum_{ij} \tilde{A}_{ij} D_{ij} u + \sum_j B_j D_j u + Cu = f + F(u)$$

where

$$F(u) = \sum_{ij} (A_{ij} - \tilde{A}_{ij})\, D_{ij}\, u. \tag{3.2}$$

The weak form of this problem is the following:

$$\lambda(u, \phi)_{0,\Omega} + \tilde{a}(u, \phi) = (f + F(u), \phi)_{0,\Omega}, \qquad \forall \phi \in H_0^1(\Omega, \mathbb{C}^N) \tag{3.3}$$

where

$$\tilde{a}(u, \phi) = \int_\Omega \left\{ \sum_{ij} \left(\tilde{A}_{ij} D_j u \,|\, D_i \phi \right) + \sum_j \left(\left[B_j - \sum_i D_i \tilde{A}_{ij} \right] D_j u \,|\, \phi \right) + (Cu \,|\, \phi) \right\} dx.$$

Let us also set

$$L = \sup_{x \in \Omega} \left[\sum_{ij} |A_{ij}(x)|^2 \right]^{1/2}, \qquad \tilde{L} = \sup_{x \in \Omega} \left[\sum_{ij} |\tilde{A}_{ij}(x)|^2 \right]^{1/2}.$$

THEOREM 3.1. *There exists $\omega_0 \geqslant 0$ such that, if* $\operatorname{Re} \lambda > \omega_0$, *then for each $f \in L^2(\Omega, \mathbb{C}^N)$ problem* (3.1) *has a unique solution u. Moreover,*

$$(|\lambda| - \omega_0)|u|_{0,\Omega} + (|\lambda| - \omega_0)^{1/2} |u|_{1,\Omega} + |u|_{2,\Omega} \leqslant c |f|_{0,\Omega} \tag{3.4}$$

where c is independent of λ and u.

Proof. By Remark 2.6 and Theorem 2.7 we conclude that, for each mollification $\tilde{A}_{ij}$ of the coefficients A_{ij} such that

$$\tilde{L} = L \qquad \text{and} \qquad \tilde{\nu} = \nu,$$

where ν and $\tilde{\nu}$ are, respectively, the ellipticity constants of the matrices A_{ij} and $\tilde{A}_{ij}$, there exists $\tilde{\omega} \geqslant 0$[1] such that, if $\operatorname{Re} \lambda > \tilde{\omega}$, then for each $u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$ the problem

$$U \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$$
$$\lambda(U, \phi)_{0,\Omega} + \tilde{a}(U, \phi) = (f + F(u), \phi)_{0,\Omega}, \qquad \forall \phi \in H_0^1(\Omega, \mathbb{C}^N)$$

[1] Here $\tilde{\omega}$ may be given by $\tilde{\omega} = \tilde{\Lambda} + \tilde{\lambda}_0$, where $\tilde{\Lambda}$ is the Gårding constant associated with the matrices $\tilde{A}_{ij}$ and

$$\tilde{\lambda}_0 = \lambda_0 + \frac{1}{\nu} \sup_{x \in \Omega} \sum_{ij} |D_i \tilde{A}_{ij}(x)|^2.$$

has a unique solution $U=\mathscr{T}_\lambda u$. Moreover

$$|\mathscr{T}_\lambda u|_{2,\Omega} \leqslant c_1 |\mathscr{T}_\lambda u|_{0,\Omega} + c_2 |f+F(u)|_{0,\Omega} \tag{3.5}$$

and

$$|\mathscr{T}_\lambda u|_{0,\Omega} \leqslant \frac{c_3}{|\lambda|-\tilde{\omega}} |f+F(u)|_{0,\Omega} \tag{3.6}$$

where c_1 is independent of λ, and where c_2 and c_3 are positive constants depending only on $L \vee \nu^{-1}$.

Now choose the mollification $\tilde{A}_{ij}$ so that

$$\sup_{x\in\Omega} \sum_{ij} [|A_{ij}(x) - \tilde{A}_{ij}(x)|]^{1/2} \leqslant 1/4c_2. \tag{3.7}$$

We claim that there exists $\omega_0 > \tilde{\omega}$ such that

$$\operatorname{Re}\lambda > \omega_0 \Rightarrow \mathscr{T}_\lambda \text{ is a contraction mapping} \tag{3.8}$$

In fact, if $v,\ w \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$, then for each $\phi \in H_0^1(\Omega, \mathbb{C}^N)$

$$\lambda(\mathscr{T}_\lambda v - \mathscr{T}_\lambda w, \phi)_{0,\Omega} + \tilde{a}(\mathscr{T}_\lambda v - \mathscr{T}_\lambda w, \phi) = (F(v-w), \phi)_{0,\Omega}$$

and so, by (3.2) and (3.7),

$$\begin{aligned}|\mathscr{T}_\lambda v - \mathscr{T}_\lambda w|_{2,\Omega} &\leqslant c_1 |\mathscr{T}_\lambda v - \mathscr{T}_\lambda w|_{0,\Omega} + c_2 |F(v-w)|_{0,\Omega} \\ &\leqslant \frac{1}{4}\left(\frac{c_1 c_3/c_2}{|\lambda|-\tilde{\omega}} + 1\right) |v-w|_{2,\Omega}.\end{aligned}$$

Therefore, recalling that $|\cdot|_{2,\Omega}$ is an equivalent norm in $H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$, in order to get (3.8) we may choose

$$\omega_0 = \tilde{\omega} + c_1 c_3/c_2. \tag{3.9}$$

The existence and uniqueness of the solution follow. To show inequality (3.4) we remark that, by (3.5), (3.6), and (3.9),

$$|u|_{2,\Omega} \leqslant \left(\frac{c_1 c_3}{|\lambda|-\tilde{\omega}} + c_2\right) |f+F(u)|_{0,\Omega} \leqslant \frac{1}{2} |u|_{2,\Omega} + 2c_2 |f|_{0,\Omega}$$

or

$$|u|_{2,\Omega} \leqslant 4c_2 |f|_{0,\Omega}. \tag{3.10}$$

Moreover

$$(|\lambda|-\tilde{\omega})|u|_{0,\Omega} \leqslant c_3 |f|_{0,\Omega} + \frac{c_3}{4c_2} |u|_{2,\Omega} \leqslant 2c_3 |f|_{0,\Omega}. \tag{3.11}$$

The estimate (3.4) easily follows from inequalities (3.10) and (3.11). ∎

4. Local Estimates

An essential step of the regularization in $\mathscr{L}^{2,\mu}$ spaces is represented by some integral inequalities, on balls and half balls, for solutions of elliptic systems with constant coefficients (see also [12, No. 2]).

Let, therefore, B_{ij}, $i, j = 1, \ldots, n$, be $N \times N$ constant complex matrices satisfying the ellipticity condition (2.7), i.e., there exists $\nu > 0$ such that for each $\xi \in \mathbb{R}^n$ and each $\eta \in \mathbb{C}^N$

$$\sum_{ij=1}^{n} \xi_i \xi_j \operatorname{Re}(B_{ij}\eta \mid \eta)_N \geqslant \nu |\xi|_n^2 \, |\eta|_N^2.$$

Repeating the proof of [12, Theorem 2.I] one can easily obtain the following result:

Theorem 4.1. *If $u \in H^1(B(r), \mathbb{C}^N)$ is a solution of the elliptic system*

$$\lambda u - \sum_{ij} B_{ij} D_{ij} u = \zeta \in \mathbb{C}^N \qquad \text{in } B(r) \tag{4.1}$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant 0$, then for each $t \in]0, 1]$ and $\sigma \in]0, r]$

$$|u|_{1,B(t\sigma)}^2 \leqslant c\, t^n |u|_{1,B(\sigma)}^2 \tag{4.2}$$

where the constant c does not depend on λ, r, and u.

We merely need to recall that the proof of Theorem 4.1 is based on the Sobolev inbedding theorem and on the Caccioppoli-type inequality (2.17), which also holds for the higher-order derivatives of u.

Remark 4.2. Using Poincaré's inequality, the estimate (4.2), and then Caccioppoli inequality (2.18) we get

$$|u - u_{B(t\sigma)}|_{0,B(t\sigma)}^2 \leqslant c\, t^{n+2} |u - u_{B(\sigma)}|_{0,B(\sigma)}^2. \tag{4.3}$$

Moreover, if $u \in H^2(B(r), \mathbb{C}^N)$, then we can differentiate system (4.1) and write (4.2) for the derivatives of u, thus obtaining

$$|u|_{2,B(t\sigma)}^2 \leqslant c\, t^n |u|_{2,B(\sigma)}^2. \tag{4.4}$$

Theorem 4.1 has an analogous version on half balls if $N = 1$ (see [12, Theorem 2.II]). A result of this kind is available even in the case of $N > 1$. The argument needed for this purpose is, however, less straightforward, as can be seen from the following proposition.

THEOREM 4.3. *Let $u \in H^2(B^+(r), \mathbb{C}^N)$ vanish on $\Gamma(r)$ and solve the elliptic system*

$$\lambda u - \sum_{ij=1}^{n} B_{ij} D_{ij} u = 0 \qquad \text{in} \quad B^+(r) \tag{4.5}$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant 0$. Then for each $\varepsilon \in]0, n[$ there exists a constant c_3, independent of λ, r, and u, such that for each $t \in]0, 1]$ and $\sigma \in]0, r]$

$$|u|^2_{1, B^+(t\sigma)} \leqslant c_3 t^{n-\varepsilon} |u|^2_{1, B^+(\sigma)}. \tag{4.6}$$

In order to pass from a problem on a half ball to a problem on a whole ball we need the following lemma:

LEMMA 4.4. *Under the hypotheses of Theorem 4.3 let us set*

$$\begin{aligned} U(x) &= u(x) && \text{if} \quad x_n \geqslant 0 \\ &= -u(x', -x_n) && \text{if} \quad x_n < 0. \end{aligned} \tag{4.7}$$

Then u belongs to $H^2(B(r), \mathbb{C}^N)$ and solves, in $B(r)$, the system

$$\lambda U - \sum_{i=1}^{n} B_{ii} D_{ii} U = F$$

where

$$F(x) = \sum_{\substack{i \neq j}}^{1, n-1} B_{ij} D_{ij} U(x) + (\operatorname{sgn} x_n) \cdot \sum_{i=1}^{n-1} (B_{in} + B_{ni})\, D_{in} U(x). \tag{4.8}$$

Proof. Let us compute the distributional derivative $D_{nn} u$ first. As u vanishes on $\Gamma(1)$, we have, for each $\phi \in C_0^\infty(B(r), \mathbb{C}^N)$,

$$\begin{aligned} &\int_{B(r)} (U \mid D_{nn}\phi)_N \, dx \\ &\quad = \int_{B^+(r)} (u \mid D_{nn}\phi)_N \, dx - \int_{B^-(r)} (u(x, -x_n) \mid D_{nn}\phi)_N \, dx \\ &\quad = -\int_{B^+(r)} (D_n u \mid D_n \phi) \, dx - \int_{B^-(r)} (D_n u(x, -x_n) \mid D_n \phi) \, dx \\ &\quad = \int_{\Gamma(r)} (D_n u \mid \phi) \, dH_{n-1} + \int_{B^+(r)} (D_{nn} u \mid \phi) \, dx \\ &\qquad - \int_{\Gamma(r)} (D_n u \mid \phi) \, dH_{n-1} - \int_{B^+(r)} (D_{nn} u(x, -x_n) \mid \phi) \, dx. \end{aligned}$$

Therefore

$$\begin{aligned} D_{nn}U(x) &= D_{nn}u(x) && \text{if} \quad x_n > 0 \\ &= -D_{nn}u(x', -x_n) && \text{if} \quad x_n < 0 \end{aligned}$$

in the distributional sense on $B(r)$.

Similarly one proves that, for each $i \neq n$,

$$\begin{aligned} D_{in}U(x) &= D_{in}u(x) && \text{if} \quad x_n > 0 \\ &= D_{in}u(x', -x) && \text{if} \quad x_n < 0 \end{aligned}$$

and, for each $i, j \neq n$,

$$\begin{aligned} D_{ij}U(x) &= D_{ij}u(x) && \text{if} \quad x_n > 0 \\ &= -D_{ij}u(x', -x_n) && \text{if} \quad x_n < 0 \end{aligned}$$

in the sense of distributions on $B(r)$.

By Remark 2.3 this means that $U \in H^2(B(r), \mathbb{C}^N)$ and the remaining part of the thesis is trivial. ∎

We can now prove a theorem which will be useful to bound the first-order derivatives of the solution u in terms of its mixed second-order derivatives:

LEMMA 4.5. *Under the hypotheses of Theorem* 4.3 *there exists a constant* c, *independent of* λ r, *and* u, *such that, for each* $t \in]0, 1]$ *and* $\sigma \in]0, r]$,

$$\int_{B^+(t\sigma/2)} \sum_i |D_i u|^2 dx \leqslant c\, t^n \int_{B^+(\sigma/2)} \sum_i |D_i u|^2 dx + c\sigma^2 \int_{B^+(\sigma/2)} \sum_{i \neq j} |D_{ij} u|^2 dx.$$

Proof. By Lemma 4.4 it results that, if U is defined as in (4.7), then $U \in H^2(B(r), \mathbb{C}^N)$ and, for each $\phi \in H_0^1(B(r), \mathbb{C}^N)$,

$$\lambda \int_{B(r)} (U \mid \phi)\, dx + \int_{B(r)} \sum_i (B_{ii} D_i U \mid D_i \phi)\, dx = \int_{B(r)} (F \mid \phi)\, dx$$

where F is defined in (4.8).

Let us split the vector U, on $B(\sigma/2)$, in the sum $v + w$, where

$$w \in H_0^1(B(\sigma/2), \mathbb{C}^N)$$

$$\lambda \int_{B(\sigma/2)} (w \mid \phi)\, dx + \int_{B(\sigma/2)} \sum_i (B_{ii} D_i w \mid D_i \phi)\, dx = \int_{B(\sigma/2)} (F \mid \phi)\, dx \tag{4.9}$$

for each $\phi \in H_0^1(B(\sigma/2), \mathbb{C}^N)$, whereas

$$v \in H^1(B(\sigma/2), \mathbb{C}^N)$$

$$\lambda \int_{B(\sigma/2)} (v \mid \phi)\, dx + \int_{B(\sigma/2)} \sum_i (B_{ii} D_i v \mid D_i \phi)\, dx = 0$$

for each $\phi \in H_0^1(B(\sigma/2), \mathbb{C}^N)$.

Choosing in (4.9) $\phi = w$, from Poincaré's inequality we conclude that

$$\int_{B(\sigma/2)} \sum_i |D_i w|^2 dx \leqslant c(v)\, \sigma^2 \int_{B(\sigma/2)} |F|^2 dx.$$

Using Theorem 4.1 to majorize v, we get, for each $t \in]0, 1]$,

$$\int_{B(t\sigma/2)} \sum_i |D_i v|^2 dx \leqslant ct^n \int_{B(\sigma/2)} \sum_i |D_i v|^2 dx.$$

From the last two inequalities we find, for each $t \in]0, 1]$,

$$\begin{aligned} &\int_{B(t\sigma/2)} \sum_i |D_i U|^2 dx \\ &\quad \leqslant 2 \int_{B(t\sigma/2)} \sum_i |D_i v|^2 dx + 2 \int_{B(\sigma/2)} \sum_i |D_i w|^2 dx \\ &\quad \leqslant ct^n \int_{B(\sigma/2)} \sum_i |D_i U|^2 dx + c\sigma^2 \int_{B(\sigma/2)} |F|^2 dx. \end{aligned}$$

The thesis of Lemma 4.5 is then a consequence of the definitions (4.7) and (4.8). ∎

Now we can show estimate (4.6).

Proof of Theorem 4.3. Assume that $n = 2$. From Lemmas 4.5 and 2.13 we have, for each $\varepsilon \in]0, 2[$, for each $t \in]0, 1]$, and each $\sigma \in]0, r]$,

$$\begin{aligned} \int_{B^+(t\sigma/2)} \sum_i |D_i u|^2 dx \leqslant ct^{2-\varepsilon} &\int_{B^+(\sigma/2)} \sum_i |D_i u|^2 dx \\ &+ c_3 (t\sigma)^{2-\varepsilon_\sigma \varepsilon} \int_{B^+(\sigma/2)} \sum_{i \neq j} |D_{ij} u|^2 dx. \end{aligned}$$

Now this estimate is, by (2.22) with $R = \sigma$ and $\rho = \sigma/2$, nothing but (4.6) for each $t \in]0, \frac{1}{2}]$ and $\sigma \in]0, r]$. The thesis is thus proved, as (4.6) is trivial for $t \in]\frac{1}{2}, 1]$.

Next, assume that $n > 2$. Again by Lemmas 4.5 and 2.13 we have, for each $t \in]0, 1]$ and each $\sigma \in]0, r]$,

$$\int_{B^+(t\sigma/2)} \sum_i |D_i u|^2 dx \leqslant ct^2 \int_{B^+(\sigma/2)} \sum_i |D_i u|^2 dx + c(t\sigma)^2 \int_{B^+(\sigma/2)} \sum_{i \neq j} |D_{ij} u|^2 dx.$$

So, recalling (2.22) we conclude that, for each $t \in]0, \frac{1}{2}[$,

$$\int_{B^+(t\sigma)} \sum_i |D_i u|^2 dx \leqslant ct^2 \int_{B^+(\sigma)} \sum_i |D_i u|^2 dx, \tag{4.10}$$

this estimate being trivial for each $t \in]\frac{1}{2}, 1]$.

For each $k = 1, ..., n-1$, $D_k u$ is a solution of system (4.5) that vanishes on $\Gamma(r)$ and so it can be estimated by (4.10): for each $\tau \in]0, 1]$ and $\sigma \in]0, r]$

$$\int_{B^+(\tau\sigma/2)} \sum_i |D_{ik} u|^2 dx \leqslant c\tau^2 \int_{B^+(\sigma/2)} \sum_i |D_{ik} u|^2 dx, \qquad k = 1, ..., n-1. \tag{4.11}$$

Now, by Lemma 4.5 we get for each $s \in]0, 1]$

$$\int_{B^+(s\tau\sigma/2)} \sum_i |D_i u|^2 dx \leqslant cs^n \int_{B^+(\tau\sigma/2)} \sum_i |D_i u|^2 dx + c(\tau\sigma)^2 \int_{B^+(\tau\sigma/2)} \sum_{i \neq j} |D_{ij} u|^2 dx$$

and so, by (4.11) and (2.22) we have

$$\int_{B^+(s\tau\sigma/2)} \sum_i |D_i u|^2 dx \leqslant cs^n \int_{B^+(\tau\sigma/2)} \sum_i |D_i u|^2 dx + c\tau^4 \int_{B^+(\sigma)} \sum_i |D_i u|^2 dx \tag{4.12}$$

for each $s, \tau \in]0, 1]$ and each $\sigma \in]0, r]$.

If $n \leqslant 4$, then by (4.12) and Lemma 2.13 we conclude that, for each $\varepsilon \in]0, n[$, each $\tau, s \in]0, 1]$, and each $\sigma \in]0, r]$,

$$\int_{B^+(s\tau\sigma/2)} \sum_i |D_i u|^2 dx \leqslant cs^{n-\varepsilon} \int_{B^+(\tau\sigma/2)} \sum_i |D_i u|^2 dx + c_\varepsilon \tau^{4-n+\varepsilon} (s\tau)^{n-\varepsilon} \int_{B^+(\sigma)} \sum_i |D_i u|^2 dx$$

which gives (4.6) taking $\tau = 1$.

On the other hand, if $n>4$, then by (4.12) and Lemma 2.13 we obtain, for each $\tau, s \in]0,1]$ and each $\sigma \in]0, r]$,

$$\int_{B^+(s\tau\sigma/2)} \sum_i |D_i u|^2 dx \leqslant cs^4 \int_{B^+(\tau\sigma/2)} \sum_i |D_i u|^2 dx + c(s\tau)^4 \int_{B^+(\sigma)} \sum_i |D_i u|^2 dx$$

and so, taking $\tau=1$,

$$\int_{B^+(s\sigma)} \sum_i |D_i u|^2 dx \leqslant cs^4 \int_{B^+(\sigma)} \sum_i |D_i u|^2 dx.$$

Repeating the argument above, (4.6) can then be achieved in a finite number of iterations. The proof of Theorem 4.3 is thus complete. ∎

Remark 4.6. Applying Poincaré's inequality, Theorem 4.3, and then the Caccioppoli-type inequality (2.21), we also get, for each $\varepsilon \in]0, n[$, each $t \in]0,1]$, and each $\sigma \in]0, r]$,

$$|u|^2_{0,B^+(t\sigma)} \leqslant c_\varepsilon t^{n+2-\varepsilon} |u|^2_{0,B^+(\sigma)}.$$

COROLLARY 4.7. *Under the hypotheses of Theorem* 4.3, *for each* $\varepsilon \in]0, n[$ *there exists a constant* c_ε, *independent of* λ, r, *and* u, *such that, for each* $t \in]0,1[$ *and each* $\sigma \in]0, r]$,

$$|u|^2_{2,B^+(t\sigma)} \leqslant c_\varepsilon t^{n-\varepsilon} |u|^2_{2,B^+(\sigma)}.$$

Proof. As has already been remarked, $D_k u$ for $k=1,\dots,n-1$ is a solution of system (4.5) that vanishes on $\Gamma(r)$.

Therefore Theorem 4.3 gives

$$\int_{B^+(t\sigma)} \sum_i |D_{ik} u|^2 dx \leqslant c_\varepsilon t^{n-\varepsilon} \int_{B^+(\sigma)} \sum_i |D_{ik} u|^2 dx, \qquad k=1,\dots,n-1 \tag{4.13}$$

for each $\varepsilon \in]0, n[$, each $t \in]0,1]$, and each $\sigma \in]0, r]$.

On the other hand, from (4.5) we get

$$D_{nn} u = B_{nn}^{-1} \left(\lambda u - \sum_{i+j<2n} B_{ij} D_{ij} u \right)$$

so that, by Remark 4.6 and (4.13), we conclude that

$$\int_{B^+(t\sigma)} |D_{nn} u|^2 dx \leqslant c_\varepsilon t^{n-\varepsilon} (|\lambda|^2 |u|^2_{0,B^+(\sigma)} + |u|^2_{2,B^+(\sigma)}). \tag{4.14}$$

The thesis follows from (4.13) and (4.14), as

$$|\lambda|^2 |u|^2_{0,B^+(\sigma)} \leqslant \sum_{ij} |B_{ij}|^2 |u|^2_{2,B^+(\sigma)}. \quad ∎$$

In order to handle Hölder norms we also need the following theorem, which generalizes Remark 4.6:

COROLLARY 4.8. *Let $u \in H^2(B^+(r), \mathbb{C}^N)$ vanish on $\Gamma(r)$ and solve the elliptic system with constant coefficients*

$$\lambda u - \sum_{ij} B_{ij} D_{ij} u = \sum_k C_k D_k u \qquad in \quad B^+(r)$$

where $\lambda \in \mathbb{C}$ and

$$\operatorname{Re} \lambda \geqslant \nu^{-1} \sum_k |C_k|^2.$$

Then for each $\varepsilon \in]0, n[$ there exists $r_\varepsilon \in]0, r]$ and $c_\varepsilon > 0$, independent of λ, r, and u, such that for each $t \in]0, 1]$ and each $\sigma \in]0, r_\varepsilon]$

$$|u|^2_{0, B^+(t\sigma)} \leqslant c_\varepsilon t^{n+2-\varepsilon} |u|^2_{0, B^+(\sigma)}. \tag{4.15}$$

Proof. It is sufficient to show that

$$|u|^2_{1, B^+(t\sigma)} \leqslant c_\varepsilon t^{n-\varepsilon} |u|^2_{1, B^+(\sigma)}, \tag{4.16}$$

in fact (4.15) then follows repeating the argument of Remark 4.6.

To prove (4.16) fix $\sigma \in]0, r]$ and split u, on $B^+(\sigma)$, in the sum $v + w$, where w is given by Lemma 2.10 and is such that

$$w \in H^2(B^+(\sigma), \mathbb{C}^N), \qquad w = 0 \quad \text{on } \Gamma(\sigma)$$

$$\lambda w - \sum_{ij} B_{ij} D_{ij} w = \sum_k C_k D_k u \qquad \text{in } B(\sigma)$$

and, by (2.15),

$$|w|_{1, B^+(\sigma)} \leqslant c\sigma |u|_{1, B^+(\sigma)},$$

whereas

$$v \in H^2(B^+(\sigma), \mathbb{C}^N), \qquad v = 0 \quad \text{on } \Gamma(\sigma)$$

$$\lambda - \sum_{ij} B_{ij} D_{ij} v = 0 \qquad \text{in } B(\sigma).$$

By Theorem 4.3 we get, for each $\varepsilon \in]0, n[$ and each $t \in]0, 1]$,

$$|v|^2_{1, B^+(t\sigma)} \leqslant c_\varepsilon t^{n-\varepsilon} |v|^2_{1, B^+(\sigma)}.$$

Therefore we have, for each $\varepsilon \in]0, n[$,

$$|u|^2_{1,B^+(t\sigma)} \leqslant 2|v|^2_{1,B^+(t\sigma)} + 2|w|^2_{1,B^+(\sigma)}$$
$$\leqslant c(t^{n-\varepsilon} + \sigma^2)|u|^2_{1,B^+(\sigma)}.$$

To obtain (4.16) one now has only to recall Lemma 2.14. ∎

The estimates which have been just proved for solutions of systems with constant coeffients can be extended to solutions of systems with continuous coefficients following a standard procedure (see [12, No. 3]).

In particular we are interested in estimates for second-order derivatives:

THEOREM 4.9. *Let A_{ij}, $i, j = 1,..., n$, be $N \times N$ complex valued matrices which we assume continuous in $\overline{B(r)}$ and elliptic in the sense of condition* (2.7).

Then for each $\varepsilon > 0$ there exists $r_\varepsilon \in]0, r]$ such that, for each $\sigma \in]0, r_\varepsilon]$ and each solution $u \in H^2(B(r), \mathbb{C}^N)$ of the system

$$\lambda u - \sum_{ij} A_{ij} D_{ij} u = 0 \qquad \text{in} \quad B(r)$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant 0$, *it results*

$$|u|^2_{2,B(t\sigma)} \leqslant c_\varepsilon t^{n-\varepsilon} |u|^2_{2,B(\sigma)}$$

for each $t \in]0, 1]$, where c_ε is independent of λ and r.

We give just a sketch of the proof, for the reader's convenience.

Fix $\sigma \in]0, r]$ and split u, on $B(\sigma)$, in the sum $v + w$, where w is given by Lemma 2.9 and is such that

$$w \in H^2 \cap H^1_0(B(\sigma), \mathbb{C}^N)$$

$$\lambda w - \sum_{ij} A_{ij}(0)\, D_{ij} w = \sum_{ij} (A_{ij} - A_{ij}(0))\, D_{ij} u \qquad \text{in } B(\sigma)$$

and

$$|w|^2_{2,B(\sigma)} \leqslant c\theta(\sigma)|u|^2_{2,B(\sigma)}, \tag{4.17}$$

where

$$\theta(\sigma) = \sup_{x \in B(\sigma)} \sum_{ij} |A_{ij}(x) - A_{ij}(0)|^2,$$

whereas

$$v \in H^2(B(\sigma), \mathbb{C}^N)$$

$$\lambda v - \sum_{ij} A_{ij}(0)\, D_{ij} v = 0 \qquad \text{in } B(\sigma).$$

By inequality (4.4) we can estimate v as follows: for each $t \in]0, 1]$

$$|v|^2_{2, B(t\sigma)} \leqslant ct^n |v|^2_{2, B(\sigma)}. \tag{4.18}$$

The thesis of Theorem 4.9 can then be obtained from (4.17) and (4.18) by Lemma 2.14. ∎

The analogous result on half balls can be proved the same way, using Lemma 2.10 instead of Lemma 2.9 and Corollary 4.7 instead of inequality (4.4):

THEOREM 4.10. *Let A_{ij}, $i, j = 1,\ldots, n$, be $N \times N$ complex valued matrices which we assume continuous in $B^+(r)$ and elliptic in the sense of condition* (2.7).

Then for each $\varepsilon > 0$ there exists $r_\varepsilon \in]0, r]$ such that, for each $\sigma \in]0, r_\varepsilon]$ and each vector $u \in H^2(B^+(r), \mathbb{C}^N)$, vanishing on $\Gamma(r)$, which solves the system

$$\lambda u - \sum_{ij} A_{ij} D_{ij} u = 0 \qquad \text{in} \quad B^+(r)$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geqslant 0$, *it results*

$$|u|^2_{2, B^+(t\sigma)} \leqslant c_\varepsilon t^{n-\varepsilon} |u|^2_{2, B^+(\sigma)}$$

for each $t \in]0, 1]$, where c_ε is independent of λ and r.

5. The Estimate in Morrey Spaces

Morrey spaces $L^{2,\mu}(\Omega, \mathbb{C}^N)$, $0 \leqslant \mu < n$, whose definition we recalled in Section 2, are known to be isomorphic to the spaces $\mathscr{L}^{2,\mu}_0(\Omega, \mathbb{C}^N)$ (see Theorem 2.1). In the present section we prove an estimate concerning the generation of analytic semigroups in these spaces. This result will be used in the next section to obtain generation in $L^p(\Omega, \mathbb{C}^N)$, $1 < p \leqslant +\infty$.

We will start our analysis from the framework developed in No. 3. Namely, let Ω be a C^2 bounded domain in $\mathbb{R}^n$ and let us consider the second-order differential operator

$$Eu = -\sum_{i,j=1}^{n} A_{ij} D_{ij} u + \sum_{j=1}^{n} B_j D_j u + Cu.$$

Here $u: \Omega \to \mathbb{C}^N$ and A_{ij}, B_j, and C are $N \times N$ complex valued matrices. Assume also that conditions (2.5), (2.6), and (2.7) hold, i.e., assume that B_j and C are bounded and measurable in Ω and that A_{ij} are continuous and elliptic in $\bar{\Omega}$.

Then there exists $\omega_0 \geqslant 0$ (given, for instance, by (3.9)) such that, if $\operatorname{Re} \lambda > \omega_0$, the Dirichlet problem

$$\begin{aligned} &u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N) \\ &(\lambda + E)\, u = f \qquad \text{in } \Omega \end{aligned} \tag{5.1}$$

has a unique solution for each $f \in L^2(\Omega, \mathbb{C}^N)$.

We will show the following theorem:

THEOREM 5.1. *If conditions* (2.5), (2.6), *and* (2.7) *hold and if* $\operatorname{Re} \lambda > \omega_0$, *then for each* $f \in L^{2,\mu}(\Omega, \mathbb{C}^N)$, $0 \leqslant \mu < n$, *the solution* u *of problem* (5.1) *belongs to* $H^2_{(\mu)}(\Omega, \mathbb{C}^N)$ *and*

$$\begin{aligned} (|\lambda| - \omega_0)\|u\|_{L^{2,\mu}(\Omega)} + (|\lambda| - \omega_0)^{1/2} \sum_i \|D_i u\|_{L^{2,\mu}(\Omega)} \\ + \sum_{ij} \|\|D_{ij} u\|_{L^{2,\mu}(\Omega)} \leqslant C\|f\|_{L^{2,\mu}(\Omega)} \end{aligned} \tag{5.2}$$

where c is independent of λ.

The proof of Theorem 5.1 follows from the local estimates proved in Section 4, adapting the techniques of [9] to the present situation.

First all let us show a lemma for solution of particular elliptic systems.

LEMMA 5.2. *Under the hypotheses of Theorem* 5.1, *for each ball* $B(x_0, r) \subset \Omega$ *and for each solution* $U \in H^2(B(x_0, r), \mathbb{C}^N)$ *of the system*

$$\lambda U - \sum_{ij} A_{ij} D_{ij} U = g \in L^{2,\mu}(B(x_0, r), \mathbb{C}^N), \qquad \operatorname{Re} \lambda \geqslant 0,$$

it results, for each $\rho \in]0, r]$,

$$|U|^2_{2, B(x_0, \rho)} \leqslant c\rho^{\mu}(\|g\|^2_{L^{2,\mu}(B(x_0, r))} + |U|^2_{2, B(x_0, r)}) \tag{5.3}$$

where c is independent of λ, x_0, and ρ.

Proof. By Lemma 2.9 we conclude that there exists $\sigma_0 \in]0, r]$ such that for each $\sigma \in]0, \sigma_0]$ we can uniquely solve the Dirichlet problem

$$w \in H^2 \cap H_0^1(B(x_0, \sigma), \mathbb{C}^N)$$

$$\lambda w - \sum_{ij} A_{ij} D_{ij} w = g \qquad \text{in} \quad B(x_0, \sigma)$$

and

$$|w|^2_{2,B(x_0,\sigma)} \leqslant c\,|g|^2_{0,B(x_0,\sigma)} \leqslant c\sigma^\mu \|g\|^2_{L^{2,\mu}(B(x_0,\sigma))} \tag{5.4}$$

where c_1 is independent of λ and σ.

Moreover, if we set $v = U - w$ on $B(x_0, \sigma)$, then

$$v \in H^2(B(x_0, \sigma), \mathbb{C}^N)$$

$$\lambda v - \sum_{ij} A_{ij} D_{ij} v = 0 \qquad \text{in} \quad B(x_0, \sigma).$$

Therefore, choosing $\varepsilon = (n-\mu)/2$, by Theorem 4.9 we deduce that there exist $r_\mu \in]0, r]$ such that, if $0 < \sigma \leqslant \sigma_0 \wedge r_\mu$, then for each $t \in]0, 1]$

$$|v|^2_{2,B(x_0,t\sigma)} \leqslant ct^{(n+\mu)/2} |v|^2_{2,B(x_0,\sigma)} \tag{5.5}$$

where c is independent of λ and σ.

From (5.4) and (5.5) we obtain, for each $\sigma \in]0, \sigma_0 \wedge r_\mu]$ and $t \in]0, 1]$,

$$\begin{aligned} |U|^2_{2,B(x_0,t\sigma)} &\leqslant 2|v|^2_{2,B(x_0,t\sigma)} + 2|w|^2_{2,B(x_0,t\sigma)} \\ &\leqslant ct^{(n+\mu)/2}|v|^2_{2,B(x_0,\sigma)} + 2|w|^2_{2,B(x_0,\sigma)} \\ &\leqslant ct^{(n+\mu)/2}|U|^2_{2,B(x_0,\sigma)} + c\sigma^\mu \|g\|^2_{L^{2,\mu}(B(x_0,\sigma))}. \end{aligned}$$

Hence, recalling Lemma 2.13 we get, for each $t \in]0, 1]$,

$$|U|^2_{2,B(x_0,t\sigma)} \leqslant ct^\mu |U|^2_{2,B(x_0,\sigma)} + c(t\sigma)^\mu \|g\|_{L^{2,\mu}(B(x_0,\sigma))}.$$

We have thus proved inequality (5.3): if $\rho \in]0, \sigma_0 \wedge r_\mu]$ it follows from the previous one choosing $\sigma = \sigma_0 \wedge r_\mu$ and $t = \rho/\sigma$, whereas it is trivial when $\rho > \sigma_0 \wedge r_\mu$. ∎

Remark 5.3. Consider a subdomain $\Omega_1 \subset\subset \Omega$ and set $d_1 = \operatorname{dist}(\Omega_1, \partial\Omega)$. For each $x_0 \in \Omega_1$ the solution u of problem (5.1) is such that

$$\lambda u - \sum_{ij} A_{ij} D_{ij} u = g = f - \sum_j B_j D_j u - Cu \qquad \textit{in} \quad B(x_0, d_1).$$

Moreover, by Remark 2.4 we conclude that $u \in H^1_{(2)}(\Omega, \mathbb{C}^N)$ and

$$\sum_i \|D_i u\|_{L^{2,2}(\Omega)} + \|u\|_{L^{2,2}(\Omega)} \leqslant c\|u\|_{2,\Omega}.$$

This inequality implies that $g \in L^{2,\mu_1}(\Omega, C^N)$ with $\mu_1 = \mu \wedge 2$ and

$$\|g\|_{L^{2,\mu_1}(\Omega)} \leqslant c\|f\|_{L^{2,\mu}(\Omega)} + c\|u\|_{2,\Omega}.$$

Therefore, using Lemma 5.2 we deduce that $u \in H^2_{(\mu_1)}(\Omega_1, \mathbb{C}^N)$ and

$$\sum_{|\alpha|=2} \|D^\alpha u\|_{L^{2,\mu_1}(\Omega_1)} \leqslant c\|f\|_{L^{2,\mu}(\Omega)} + c\|u\|_{2,\Omega},$$

where c_1 is independent of λ.

This procedure may be easily iterated for a finite number of times, thus leading to the conclusion that $u \in H^2_{(\mu)}(\Omega', \mathbb{C}^N)$ for each open subdomain $\Omega' \subset\subset \Omega$ and

$$\sum_{|\alpha|=2} \|D^\alpha u\|_{L^{2,\mu}(\Omega')} \leqslant c\|f\|_{L^{2,\mu}(\Omega)} + c\|u\|_{2,\Omega}. \tag{5.6}$$

We now need to estimate the second-order derivatives of u near the boundary of Ω. To begin with, let us show the following theorem for the unit half ball:

LEMMA 5.4. *Let A_{ij} be continuous elliptic matrices, defined in $\overline{B^+(1)}$, and let λ be a complex number with* $\operatorname{Re} \lambda \geqslant 0$. *Then for each solution $U \in H^2(B^+(1), \mathbb{C}^N)$ of the system*

$$\lambda U - \sum_{i,j} A_{ij} D_{ij} U = g \in L^{2,\mu}(B^+(1), \mathbb{C}^N), \tag{5.7}$$

such that

$$U = 0 \quad \text{on} \quad \Gamma(1)$$

and for each $\rho \in\ ]0, 1]$, it results

$$|U|^2_{2,B^+(\rho)} \leqslant c\rho^\mu(\|g\|^2_{L^{2,\mu}(B^+(1))} + |U|^2_{2,B^+(1)}) \tag{5.8}$$

where c is independent of λ and ρ.

Proof. By Lemma 2.10 we conclude that there exists $\sigma_0 \in\ ]0, 1]$ such that, for each $\sigma \in\ ]0, \sigma_0]$, we can solve the problem

$$w \in H^2(B^+(\sigma), C^N), \qquad w = 0 \quad \text{on } \Gamma(\sigma)$$

$$\lambda w - \sum_{i,j} A_{ij} D_{ij} w = g \qquad\qquad \text{in } B^+(\sigma)$$

in such a way that

$$|w|^2_{2,B^+(\sigma)} \leqslant c|g|^2_{0,B^+(\sigma)} \leqslant c\sigma^\mu \|g\|^2_{L^{2,\mu}(B^+(1))}. \tag{5.9}$$

Then, if we set $v = U - w$, we obtain that

$$v \in H^2(B^+(\sigma), C^N), \qquad v = 0 \quad \text{on } \Gamma(\sigma)$$

$$\lambda v - \sum_{ij} A_{ij} D_{ij} v = 0 \qquad\qquad \text{in } B^+(\sigma).$$

Therefore v can be estimated by recalling Corollary 4.7: for each $t \in]0, 1]$

$$|v|^2_{2,B^+(t\sigma)} \leqslant c\, t^{(n+\mu)/2} |v|^2_{2,B^+(\sigma)} \tag{5.10}$$

where c is independent of λ and σ.

Inequality (5.8) follows from (5.9) and (5.10), recalling Lemma 2.13, as in the final part of the proof of Lemma 5.2. ▮

Remark 5.5. If the system (5.7) is not reduced to the leading part, then the conclusion of Lemma 5.4 must be slightly modified. Suppose that $U \in H^2(B^+(1), \mathbb{C}^N)$ is a solution, vanishing on $\Gamma(1)$, of the elliptic system

$$\lambda U - \sum_{i,j} A_{ij} D_{ij} U + \sum_j B_j D_j U + C = g.$$

Here the matrices B_j and C are, as usual, bounded and measurable in $B^+(1)$.

By Remark 2.4 we have $u \in H^1_{(2)}(B^+(1), \mathbb{C}^N)$ and

$$\sum_i \|D_i U\|_{L^{2,2}(B^+(1))} + \|U\|_{L^{2,2}(B^+(1))} \leqslant c \|U\|_{2,B^+(1)}.$$

Therefore, if we set

$$G = g - \sum_j B_j D_j U - CU,$$

then we get $G \in L^{2,\mu_1}(B^+(1), \mathbb{C}^N)$ with $\mu_1 = \mu \wedge 2$ and

$$\|G\|_{L^{2,\mu_1}(B^+(1))} \leqslant c \|g\|_{L^{2,\mu}(B^+(1))} + c \|U\|_{2,B^+(1)}.$$

By Lemma 5.4 it follows, for each $\rho \in]0, 1]$,

$$|U|^2_{2,B^+(\rho)} \leqslant C \rho^{\mu_1} (\|g\|^2_{L^{2,\mu}(B^+(1))} + \|U\|^2_{2,B^+(1)})$$

where c is independent of λ and ρ.

The last inequality and Lemma 5.2 imply, by a standard argument (see [6, No. 13]), that U belongs to $H^2_{(\mu_1)}(B^+(r), \mathbb{C}^N)$ for each $r \in]0, 1[$ and

$$\sum_{|\alpha|=2} \|D^\alpha U\|_{L^{2,\mu_1}(B^+(r))} \leqslant c \|g\|_{L^{2,\mu}(B^+(1))} + c \|U\|_{2,B^+(1)}$$

with c independent of λ.

Since this fact now gives $u \in H^1_{(2+\mu_1)}(B^+(r), \mathbb{C}^N)$ and so $G \in L^{2,\mu_2}(B^+(1), \mathbb{C}^N)$ with $\mu_2 = \mu \wedge (2 + \mu_1)$, the previous procedure may be iterated for a finite number of times to show that, for each $r \in]0, 1[$, U belongs to $H^2_{(\mu)}(B^+(r), \mathbb{C}^N)$ and

$$\sum_{|\alpha|=2} \|DU\|_{L^{2,\mu}(B^+(r))} \leqslant c \|g\|_{L^{2,\mu}(B^+(1))} + c \|U\|_{2,B^+(1)} \tag{5.11}$$

with c independent of λ. ▮

We are now ready to prove the main result of this section.

Proof of Theorem 5.1. We will demonstrate inequality (5.2) in two steps. The first step is to show it for the second-order derivatives of u and the second is to obtain the complete estimate.

As $\partial\Omega$ is of class C^2, there exists a finite open cover $\{\mathscr{U}_1,..., \mathscr{U}_m\}$ of $\partial\Omega$ and a corresponding set $\{\Phi_1,..., \Phi_m\}$ of one-to-one transformations taking $\mathscr{U}_k$ onto $B(1)$ and satisfying conditions (2.1) and (2.2).

For each $k=1,..., m$ let us set $U^k(x)=u(\Phi_k^{-1}(x))$ for $x\in B^+(1)$. Then U^k vanishes on $\Gamma(1)$ and solves in $B^+(1)$ the transformed system

$$\lambda U^k-\sum_{ij} A^k_{ij}D_{ij}U^k+\sum_j B^k_j\, D_jU^k+C^kU^k=f^k$$

$$\sum_{|\alpha|=2}\|D^\alpha U^k\|_{L^{2,\mu}(B^+(r))}\leqslant c\|f^k\|_{L^{2,\mu}(B^+(1))}+c\|U^k\|_{2,B^+(1)} \tag{5.12}$$

with c independent of λ.

Now choose r so that $\{\Phi_1^{-1}(B(r)),..., \Phi_m^{-1}(B(r))\}$ still covers $\partial\Omega$. By the inequalities (5.12) for $k=1,..., m$ and by inequality (5.6) we conclude that $u\in H^2_{(\mu)}(\Omega, \mathbb{C}^N)$ and

$$\sum_{|\alpha|=2}\|D^\alpha u\|_{L^{2,\mu}(\Omega)}\leqslant c\|f\|_{L^{2,\mu}(\Omega)}+c\|u\|_{2,\Omega}$$

with c independent of λ.

This inequality can be bettered; indeed, being $\operatorname{Re}\lambda>\omega_0$ (ω_0 defined in (3.9)), inequalities (3.10) and (3.11) give

$$|u|_{2,\Omega}+|u|_{0,\Omega}\leqslant c|f|_{0,\Omega}$$

with c independent of λ.

The last estimate allows us to bound the term $\|u\|_{2,\Omega}$ and so to get the first step of the proof:

$$\sum_{|\alpha|=2}\|D^\alpha u\|_{L^{2,\mu}(\Omega)}\leqslant c\|f\|_{L^{2,\mu}(\Omega)} \tag{5.13}$$

where c is independent of λ.

Next, directly from system (5.1), we obtain

$$\begin{aligned}|\lambda|\ \|u\|_{L^{2,\mu}(\Omega)}&=\left\|\sum_{ij}A_{ij}D_{ij}u-\sum_j B_jD_ju-Cu+f\right\|_{L^{2,\mu}(\Omega)}\\&\leqslant\|f\|_{L^{2,\mu}(\Omega)}+a\sum_{ij}\|D_{ij}u\|_{L^{2,\mu}(\Omega)}\\&\quad+\beta\sum_j\|D_ju\|_{L^{2,\mu}(\Omega)}+\gamma\|u\|_{L^{2,\mu}(\Omega)}\end{aligned}$$

where

$$\alpha = \sup_{x \in \Omega} \left[\sum_{ij} |A_{ij}(x)|^2 \right]^{1/2},$$

$$\beta = \sup_{x \in \Omega} \left[\sum_{ij} |B_j(x)|^2 \right]^{1/2}, \qquad \gamma = \sup_{x \in \Omega} |C(x)|.$$

Then, form (5.13) and from the interpolation inequality (2.3) it follows that

$$|\lambda| \, \|u\|_{L^{2,\mu}(\Omega)} \leqslant c \|f\|_{L^{2,\mu}(\Omega)} + (\gamma + \beta^2/\nu) \|u\|_{L^{2,\mu}(\Omega)} \tag{5.14}$$

with c independent of λ.

Finally, recalling (3.9), it results

$$\gamma + \beta^2/\nu < \tilde{\omega} < \omega_0 \tag{5.15}$$

and so the thesis of Theorem 5.1 follows from (5.13) and (5.14), after another application of Lemma 2.5. ∎

6. The Uniform and L^p Estimates

Generation of analytic semigroups in the uniform topology was proved by Stewart [17, 18], if $N = 1$, for general elliptic operators under general boundary conditions. His method consisted in bounding suitable localizations of the solutions by the L^p estimates due to Agmon [3].

In the present section we will obtain both the uniform and the L^p estimates, in the case of $N \geqslant 1$, for a second-order elliptic operator

$$Eu = -\sum_{ij=1}^{n} A_{ij} D_{ij} u + \sum_{j=1}^{n} B_j D_j u + Cu, \qquad u\colon \Omega \to \mathbb{C}^N,$$

under Dirichlet boundary conditions.

Here Ω is, as usual, an open bounded domain in $\mathbb{R}^n$ with boundary of class C^2 and A_{ij}, B_j, and C are $N \times N$ complex matrices satisfying conditions (2.5), (2.6), and (2.7), i.e., A_{ij} are continuous and elliptic in $\bar{\Omega}$, whereas B_j and C are bounded and measurable.

The logical sequence is reversed, if compared with Stewart's method: generation in $C^0(\bar{\Omega}, \mathbb{C}^N)$ turns out to be a consequence of the Morrey space estimates and then generation in $L^p(\Omega, \mathbb{C}^N)$, $1 < p < +\infty$, follows by interpolation techniques.

Before stating the main result of this section we recall that, by virtue of

Theorem 3.1, there exists $\omega_0 \geqslant 0$ such that, if $\operatorname{Re}\lambda > \omega_0$, then for each $f \in L^2(\Omega, \mathbb{C}^N)$ the Dirichlet problem

$$\begin{aligned} &u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N) \\ &(\lambda + E)\, u = f \qquad \text{in } \Omega \end{aligned} \tag{6.1}$$

has a unique solution. Moreover, by Theorem 5.1 we have that, if $f \in L^{2,\mu}(\Omega, \mathbb{C}^N)$, $0 \leqslant \mu < n$, then $u \in H^2_{(\mu)}(\Omega, \mathbb{C}^N)$ and inequality (5.2) holds. Generation in the uniform topology derives from the following proposition:

THEOREM 6.1. *Under the hypotheses* (2.5), (2.6), *and* (2.7), *there exists* $\omega_1 \geqslant \omega_0$ *such that, if* $\operatorname{Re}\lambda > \omega_1$ *and* $f \in L^\infty(\Omega, \mathbb{C}^N)$, *then the solution* u *of problem* (6.1) *belongs to* $H^{1,\infty}(\Omega, \mathbb{C}^N)$ *and*

$$(|\lambda| - \omega_1)|u|_{0,\infty,\Omega} + (|\lambda| - \omega_1)^{1/2}|u|_{1,\infty,\Omega} \leqslant c|f|_{0,\infty,\Omega} \tag{6.2}$$

where c *is independent of* λ.

Before proving Theorem 6.1, let us show the following properties of the $H^1_{(\mu)}$ spaces:

LEMMA 6.2. *Let* Ω_0 *be a convex bounded domain in* $\mathbb{R}^n$ *of diameter* d_0. *Then for each* $\varepsilon > 0$ *there exists a constant* c_ε *such that, for each* $U \in H^1_{(\mu)}(\Omega_0, \mathbb{C}^N)$, $n-2 < \mu < n$,

$$\begin{aligned} d_0^{-1}\,|U|_{0,\infty,\Omega_0} \leqslant{}& \varepsilon d_0^{(\mu-n)/2} \sum_i \|D_i U\|_{L^{2,\mu}(\Omega_0)} \\ &+ c_\varepsilon d_0^{(\mu-n-2)/2}\,\|U\|_{L^{2,\mu}(\Omega_0)} \end{aligned} \tag{6.3}$$

where c_ε *does not vary under homothetical transformations. Moreover, if* U *vanishes at a point of* Ω_0, *then*

$$d_0^{-1}\,|U|_{0,\infty,\Omega_0} \leqslant c\, d_0^{(\mu-n)/2} \sum_i \|D_i U\|_{L^{2,\mu}(\Omega_0)} \tag{6.4}$$

where c *does not vary under homothetical transformations.*

Proof. By Remark 2.4 and Theorem 2.1 we conclude that, for $n-2 < \mu < n$, $H^1_{(\mu)}(\Omega_0, \mathbb{C}^N) \subset C^{0,\alpha}(\bar{\Omega}_{,0}, \mathbb{C}^N)$ with $\alpha = 1 + (\mu - n)/2$ and

$$[U]_{C^{0,\alpha}(\bar{\Omega}_0)} \leqslant c \sum_i \|D_i U\|_{L^{2,\mu}(\Omega_0)}. \tag{6.5}$$

Therefore, taking into account Remark 2.2 as well,

$$H^1_{(\mu)}(\Omega_0, \mathbb{C}^N) \subset L^\infty(\Omega_0, \mathbb{C}^N) \subset L^{2,\mu}(\Omega_0, \mathbb{C}^N),$$

the former inclusion being compact and the latter being continuous. This fact implies the existence, for each $\varepsilon > 0$, of a constant k_ε such that

$$|U|_{0,\infty,\Omega_0} \leqslant \varepsilon \sum_i \|D_i U\|_{L^{2,\mu}(\Omega_0)} + k_\varepsilon \|U\|_{L^{2,\mu}(\Omega_0)}$$

(see [14, Lemma 5.1, chap. 1]). Inequality (6.3) then follows by a standard homothetical argument.

Inequality (6.4) may be easily deduced from (6.5), since

$$|U|_{0,\infty,\Omega_0} \leqslant d_0^\alpha [U]_{C^{0,\alpha}(\bar{\Omega}_0)}$$

for each U that vanishes at a point of $\bar{\Omega}_0$. ∎

Proof of Theorem 6.1. The technique is similar to that of Stewart [17, 18] with the difference that, now, Morrey space estimates take the place of L^p estimates.

The first step is a suitable localization of problem (6.1).

For $x \in \bar{\Omega}$ and $r > 0$, let $\Omega(x, r)$ denote the set $\Omega \cap B(x, r)$. We remark that, because of the regularity of the boundary of Ω, the set $\Omega(x, r)$ may be assumed convex, at least if r does not exceed a given number r_0, independent of x.

Let θ be a standard cutoff function:

$$\theta \in C_0^\infty(B(x, r)), 0 \leqslant \theta \leqslant 1, \theta = 1 \text{ on } B(x, r/2), |D^\alpha \theta| \leqslant c r^{-|\alpha|} \quad \text{for } |\alpha| \leqslant 2. \tag{6.6}$$

If $\operatorname{Re} \lambda > \omega_0$ and if u is the solution of problem (6.1), then

$$\theta u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$$
$$(\lambda + E)(\theta u) = g$$

where

$$g = \theta f - \sum_{ij} A_{ij}(D_i u D_j \theta + D_j u D_i \theta + u D_{ij} \theta) + \sum_j B_j u D_j \theta.$$

Therefore, assuming $r \leqslant 1$, by (6.6) we obtain

$$\|g\|_{L^{2,\mu}(\Omega(x,r))} \leqslant \| \|f\|_{L^{2,\mu}(\Omega(x,r))} + k_1 \left(r^{-2} \|u\|_{L^{2,\mu}(\Omega(x,r))} + r^{-1} \sum_i \|D_i u\|_{L^{2,\mu}(\Omega(x,r))} \right)$$

where μ is a fixed number with $n - 2 < \mu < n$.

Thus, recalling Remark 2.2,

$$\|g\|_{L^{2,\mu}(\Omega(x,r))} \leqslant k_2 r^{(n-\mu)/2}(|f|_{0,\infty,\Omega}+r^{-2}|u|_{0,\infty,\Omega}+r^{-1}|u|_{1,\infty,\Omega}) \tag{6.7}$$

where k_2 is independent of r.

Next, we use the estimate (5.2) for θu:

$$\sum_{ij}\|D_{ij}(\theta u)\|_{L^{2,\mu}(\Omega(x,r))}+(|\lambda|-\omega_0)^{1/2}\sum_i \|D_i(\theta u)\|_{L^{2,\mu}(\Omega(x,r))} \leqslant k_3\|g\|_{L^{2,\mu}(\Omega(x,r))} \tag{6.8}$$

where k_3 is independent of λ and r.

Now, from inequalities (6.3) and (6.4) we get, for each $\varepsilon>0$,

$$\begin{aligned} r^{-2}|\theta u|_{0,\infty,\Omega(x,r)}+r^{-1}|\theta u|_{1,\infty,\Omega(x,r)} &\leqslant \varepsilon r^{(\mu-n)/2}\sum_{ij}\|D_{ij}(\theta u)\|_{L^{2,\mu}(\Omega(x,r))} \\ &\quad + c_\varepsilon r^{(\mu-n-2)/2}\sum_i \|D_i(\theta u)\|_{L^{2,\mu}(\Omega(x,r))} \end{aligned} \tag{6.9}$$

where c_ε is independent of r.

Then, from (6.9), (6.8), and (6.7) we conclude that

$$\begin{aligned} r^{-2}|\theta u|_{0,\infty,\Omega(x,r)}+r^{-1}|\theta u|_{1,\infty,\Omega(x,r)} &\leqslant k_2k_3\left(\varepsilon+\frac{c_\varepsilon}{r(|\lambda|-\omega_0)^{1/2}}\right) \\ &\quad \times(|f|_{0,\infty,\Omega}+r^{-2}|u|_{0,\infty,\Omega}+r^{-1}|u|_{1,\infty,\Omega}). \end{aligned} \tag{6.10}$$

Finally, the thesis of Theorem 6.1 can be achieved by a proper choice of ε and r. Indeed, let us set

$$\varepsilon=1/8k_1k_2, \qquad K=8c_\varepsilon k_2k_3, \qquad r=K(|\lambda|-\omega_0)^{-1/2} \tag{6.11}$$

and, to make sure that $r\leqslant 1$,

$$\omega_1=\omega_0+K. \tag{6.12}$$

Using inequality (6.10) twice, once to estimate the maximum of $|u|$ and then to estimate the maximum of $(\sum_i |D_iu|^2)^{1/2}$, we conclude that, if $\operatorname{Re}\lambda>\omega_1$, then

$$\begin{aligned} (|\lambda|-\omega_0)\,K^{-2}|u|_{0,\infty,\Omega}+(|\lambda|-\omega_0)^{1/2}K^{-1}|u|_{1,\infty,\Omega} &\leqslant \tfrac12[|f|_{0,\infty,\Omega}+(|\lambda|-\omega_0)\,K^{-2}|u|_{0,\infty,\Omega} \\ &\quad +(|\lambda|-\omega_0)^{1/2}K^{-1}|u|_{1,\infty,\Omega}] \end{aligned} \tag{6.13}$$

whence inequality (6.2) easily follows. ∎

Remark 6.3. The estimation of ω_1 given by (6.12) could be bettered. It would be easy to prove that the thesis of Theorem 6.1 holds with ω_1 replaced by ω_0.

Remark 6.4. From inequalities (6.7) and (6.8) we also have

$$\sum_{ij} \|D_{ij}u\|_{L^{2,\mu}(\Omega(x,r/2))} \leqslant c r^{(n-\mu)/2}(|f|_{0,\infty,\Omega} + r^{-2}|u|_{0,\infty,\Omega} + r^{-1}|u|_{1,\infty,\Omega})$$

for each $x \in \bar{\Omega}$ and each $r \in]0, 1]$.

Hence, choosing r as in (6.11), by inequality (6.13) we conclude that

$$(|\lambda| - \omega_0)^{(n-\mu)/2} \sup_{x \in \bar{\Omega}} \sum_{ij} \|D_{ij}u\|_{L^{2,\mu}(\Omega(x,r/2))} \leqslant c|f|_{0,\infty,\Omega}$$

where c is independent of λ. This estimation is the Morrey space version of an inequality involving L^p norms, which was obtained by Stewart in [17, 18].

Remark 6.5. Another consequence of inequality (6.2) is the following estimate

$$(|\lambda| - \omega_1)^{1-\alpha/2}\|u\|_{C^{0,\mu}(\bar{\Omega})} \leqslant c|f|_{0,\infty,\Omega}$$

which is due to von Wahl [20, 21], who also showed that it cannot be bettered assuming $f \in C^{0,\alpha}(\bar{\Omega}, (\mathbb{C}^N))$.

Remark 6.6. By interpolating between the situations of Theorems 3.1 and 6.1 we get the following L^p estimate: if $\operatorname{Re} \lambda > \omega_1$ *and* $f \in L^p(\Omega, \mathbb{C}^N)$, $2 \leqslant p \leqslant +\infty$, *then*

$$(|\lambda| - \omega_1)|u|_{0,p,\Omega} + (|\lambda| - \omega_1)^{1/2}|u|_{1,p,\Omega} \leqslant c|f|_{0,p,\Omega} \tag{6.14}$$

where c is independent of λ.

Inequality (6.14) may be completed using Lemma 2.15, in order to obtain the following L^p generation result:

THEOREM 6.7. *Under the hypotheses* (2.5), (2.6), *and* (2.7) *there exists* $\omega_2 \geqslant 0$ *such that, if* $\operatorname{Re} \lambda > \omega_2$ *and* $f \in L^p(\Omega, \mathbb{C}^N)$, $2 \leqslant p \leqslant +\infty$, *then the system* $(\lambda + E)u = f$ *has a unique solution in the space* $H^{2,p} \cap H_0^{1,p}(\Omega, \mathbb{C}^N)$. *Moreover,*

$$(|\lambda| - \omega_2)|u|_{0,p,\Omega} + (|\lambda| - \omega_2)^{1/2}|u|_{1,p,\Omega} + |u|_{2,p,\Omega} \leqslant c|f|_{0,p,\Omega} \tag{6.15}$$

where c is independent of λ.

Proof. We already know that, if $\operatorname{Re}\lambda > \omega_1$, then the problem

$$u \in H^2 \cap H_0^{1,p}(\Omega, \mathbb{C}^N), \qquad 2 \leqslant p < +\infty$$
$$(\lambda + E)\, u = f L^p(\Omega, \mathbb{C}^N)$$

has a unique solution. Since (6.14) holds as well, the only thing to prove is that $u \in H^{2,p}(\Omega)$ and

$$|u|_{2,p,\Omega} \leqslant c|f|_{0,p,\Omega}.$$

This will be done by a covering argument.

First of all, let us consider a ball $B(x, r) \subset \Omega$. If θ is chosen as in (6.6), then

$$\theta u \in H^2 \cap H_0^{1,p}(B(x, r), \mathbb{C}^N)$$
$$-\sum_{ij} A_{ij} D_{ij}(\theta u) = G$$

where

$$G = -\sum_{ij} A_{ij}(u D_{ij}\theta + D_j u D_i \theta + D_i u D_j \theta) + \theta(f - \lambda u - \sum_j B_j D_j u - Cu)$$

and so, by Remark 6.6, $G \in L^p(B(x, r), \mathbb{C}^N)$.

Moreover, if

$$\operatorname{Re}\lambda > \omega_1 + 1 = \omega_2,$$

then (6.14) yields

$$|G|_{0,p,B(x,r)} \leqslant c|f|_{0,p,\Omega}$$

where c is independent of λ.

Therefore, applying Lemma 2.15

$$|\theta u|_{2,p,B(x,r)} \leqslant c|f|_{0,p,\Omega} \tag{6.16}$$

at least if r does not exceed a given number r_0, independent of x.

Thus, for any subdomain $\Omega_0 \subset\subset \Omega$

$$|u|_{2,p,\Omega_0} \leqslant c|f|_{0,p,\Omega}$$

where c is independent of λ.

It now remains to estimate u near the boundary.

Let $\{U_1, \ldots, U_m\}$ be a finite open cover of $\partial\Omega$ and let $\{\Phi_1, \ldots, \Phi_m\}$ be a set of one-to-one transformations, Φ_k taking U_k onto $B(1)$ for $k = 1, \ldots, m$ and satisfying conditions (2.1) and (2.2). If we set $U^k(x) = u(\Phi_k^{-1}(x))$ for $x \in B^+(1)$, then U^k solves the transformed system

$$U^k \in H^2 \cap H^{1,p}(B^+(1), \mathbb{C}^N), \qquad U^k = 0 \text{ on } \Gamma(1)$$
$$\lambda U^k - \sum_{ij} A_{ij}^k D_{ij} U^k + \sum B_j^k D_j U^k + C^k U^k = f^k \qquad \text{in } B^+(1).$$

Let us consider an open domain A, with a smooth boundary and such that $B^+(1/2) \subset A \subset B^+(1)$.

For $r \in]0, 1]$ let $A(r)$ denote the image of A under the homothetical transformation $x \to rx$ and let θ be a usual cutoff function, equal to 1 on $B(r/4)$ and to 0 out of $B(r/2)$.

Now

$$\theta U^k \in H^2 \cap H_0^{1,p}(A(r), \mathbb{C}^N)$$

and so, repeating the procedure that led to inequality (6.16) (see also Remark 2.16), we conclude that, if r is sufficiently small, then

$$|U^k|_{2,p,B^+(r/4)} \leqslant c|f|_{0,p,\Omega}$$

or

$$|u|_{2,p,\Phi_k^{-1}(B^+(r/4))} \leqslant c|f|_{0,p,\Omega}$$

where $k = 1,..., m$.

Since we may assume, without loss of generality, that $\{\Phi_1^{-1}(B(r/4)),..., \Phi_m^{-1}(B(r/4))\}$ still covers $\partial\Omega$, the proof of Theorem 6.7 is thus complete. ∎

Remark 6.8. The case of $1 < p < 2$ can be treated by a standard duality argument. We can thus state the existence of a positive number ω_3 such that, *if* $\operatorname{Re} \lambda > \omega_3$ *and* $f \in L^p(\Omega, \mathbb{C}^N)$, $1 < p < +\infty$, *then the system*

$$(\lambda + E) u = f$$

has a unique solution in the space $H^{2,p} \cap H_0^{1,p}(\Omega, \mathbb{C}^N)$. *Inequality* (6.15) *holds as well.*

Moreover, a generation result in $L^1(\Omega, \mathbb{C}^N)$ may also be deduced from Theorem 6.1 by the argument of Pazy [15, No. 7.3].

7. The Estimate in Hölder Spaces

In this section we shall strengthen the general assumptions we worked with so far.

Namely, let us fix $\alpha \in]0, 1[$ and let Ω be an open bounded domain in $\mathbb{R}^n$, with boundary of class $C^{2,\alpha}$. This means that, in condition (2.1), Φ_j and Φ_j^{-1} have Hölder continuous second-order derivatives with exponent α.

We shall consider second-order elliptic operators of the form

$$Eu = -\sum_{ij=1}^{n} A_{ij} D_{ij} u + \sum_{j=1}^{n} B_j D_j u + Cu, \qquad u\colon \Omega \to \mathbb{C}^N,$$

where

$$\begin{aligned}&A_{ij},\ B_j, \text{ and } C \text{ are } N\times N \text{ complex valued matrices,}\\ &\text{Hölder continuous in } \bar{\Omega} \text{ with exponent } \alpha.\end{aligned} \tag{7.1}$$

We already know the existence of a positive number ω_0 such that, if λ is a complex number with $\operatorname{Re}\lambda > \omega_0$, then for each $f \in L^2(\Omega, \mathbb{C}^N)$ the Dirichlet problem

$$\begin{aligned}&u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)\\ &(\lambda + E)\, u = f\end{aligned} \tag{7.2}$$

has a unique solution.

We recall that the Banach space $\mathscr{L}_0^{2,n+2\alpha}(\Omega, \mathbb{C}^N)$ consists of the vectors that are Hölder continuous in $\bar{\Omega}$ with exponent α and vanish on $\partial\Omega$ (see No. 2).

In this section we want to prove the following result

THEOREM 7.1. *Under the hypotheses* (7.1) *and* (7.2) *there exists a positive number* ω *such that, if* $\operatorname{Re}\lambda > \omega$, *then for each* $f \in \mathscr{L}_0^{2,n+2\alpha}(\Omega, \mathbb{C}^N)$ *the solution* u *of* (7.2) *belongs to* $H^2_{(n+2\alpha)}(\Omega, \mathbb{C}^N)$ *and*

$$\begin{aligned}(|\lambda| - \omega)\|u\|_{C^{0,\alpha}(\bar{\Omega})} &+ (|\lambda| - \omega)^{1/2} \sum_i \|D_i u\|_{C^{0,\alpha}(\bar{\Omega})}\\ &+ \sum_{|\beta|=2} \|D^\beta u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant c\|f\|_{C^{0,\alpha}(\bar{\Omega})}\end{aligned} \tag{7.3}$$

where c *is independent of* λ.

Remark 7.2. Now the operator's domain is not dense in $\mathscr{L}_0^{2,n+2\alpha}(\Omega, \mathbb{C}^N)$. Therefore Theorem 7.1 yields a generation result in the closure of such a domain, which consists of the elements $v \in \mathscr{L}_0^{2,n+2\alpha}(\Omega, \mathbb{C}^N)$ satisfying the relation

$$\lim_{r \to 0} \sup_{x, y \in \bar{\Omega}, 0 < |x-y| \leqslant r} \frac{|v(x) - v(y)|_N}{|x-y|_n^\alpha} = 0.$$

The proof of Theorem 7.1 is obtained as a consequence of the local estimates proved in Section 4, as always happens for the regularization in the $\mathscr{L}^{2,\mu}$ spaces (see [6, 9–12]).

To begin with, let us estimate the Hölder norms of certain classes of solutions to the system

$$(\lambda + E)\, U = g \qquad \text{on a ball} \quad B(x_0, r) \subset \Omega, \tag{7.4}$$

where we assume $\operatorname{Re} \lambda > 0$.

LEMMA 7.3. *Under conditions* (7.1) *and* (2.7), *let g be Hölder* continuous *in $B(x_0, r)$ with exponent α. Then, for each solution $U \in H^2_{(n-\alpha)}(B(x_0, r), \mathbb{C}^N)$ of system* (7.4) *and for each $t \in]0, 1]$*

$$\begin{aligned} |U - U_{B(x_0, tr)}|^2_{0, B(x_0, tr)} \leqslant c \Bigg\{ |\lambda|^{-2} \Bigg([g]^2_{C^{0,\alpha}(\overline{B(x_0, r)})} \\ + \sum_{|\beta| \leqslant 2} \|D^\beta U\|^2_{L^{2, n-\alpha}(Bx_0, r))} \Bigg) r^{n+\alpha} \\ + |U|^2_{0, B(x_0, r)} \Bigg\} t^{n+\alpha} \end{aligned} \tag{7.5}$$

where c is independent of λ and r.

Moreover, if $U \in H^{2,\infty}(B(x_0, r), \mathbb{C}^N)$, then, for each $t \in]0, 1]$,

$$\begin{aligned} |U - U_{B(x_0, tr)}|^2_{0, B(x_0, tr)} \leqslant c \Bigg\{ |\lambda|^{-2} \Bigg([g]^2_{C^{0,\alpha}(\overline{B(x_0, r)})} + \sum_{j=0}^{2} |U|^2_{j, \infty, B(x_0, r)} \Bigg) \\ \times r^{n+2\alpha} + |U|^2_{0, B(x_0, r)} \Bigg\} t^{n+2\alpha} \end{aligned} \tag{7.6}$$

where c is independent of λ and r.

Proof. For the sake of simplicity let us set $B(r) = B(x_0, r)$. We show inequality (7.5) first.

Let us solve the following Dirichlet problem for an arbitrary $\sigma \in]0, r]$

$$\begin{gathered} w \in H^2 \cap H^1_0(B(\sigma), \mathbb{C}^N) \\ \lambda w - \sum_{ij} A_{ij}(x_0)\, D_{ij} w = G \end{gathered}$$

where

$$\begin{aligned} G = g - g(x_0) + \sum_{ij} (A_{ij} - A_{ij}(x_0))\, D_{ij} U - \sum_{j} (B_j D_j U - B_j(x_0)) \\ -(CU - C(x_0)\, U(x_0)). \end{aligned}$$

Then, by inequality (2.13) we get

$$|\lambda|\ |w|_{0,B(\sigma)} \leqslant c|G|_{0,B(\sigma)} \tag{7.7}$$

where c is independent of λ and σ.

On the other hand, recalling Remark 2.4 and Theorem 2.1 we have

$$U \in H^2_{(n-\alpha)}(B(r), \mathbb{C}^N) \Rightarrow U, D_i U \in C^{2,n+2-\alpha}(B(r), \mathbb{C}^N) \subset C^{0,\alpha/2}(\overline{B(r)}, \mathbb{C}^N)$$

and

$$\|U\|_{C^{0,\alpha/2}(\overline{B(r)})} + \sum_j \|D_j U\|_{C^{0,\alpha/2}(\overline{B(r)})} \leqslant c \sum_{|\beta| \leqslant 2} \|D^\beta U\|_{L^{2,n-\alpha}(B(r))}. \tag{7.8}$$

Let us set

$$K_1 = [g]_{C^{0,\alpha}(\overline{B(r)})} + \sum_{|\beta| \leqslant 2} \|D^\beta U\|_{L^{2,n-\alpha}(B(r))}. \tag{7.9}$$

Then, recalling condition (7.1), by (7.8) we get

$$\begin{aligned} |G|_{0,B(\sigma)} &\leqslant c\sigma^{\alpha+n/2}[g]_{C^{0,\alpha}(\overline{B(r)})} + c\sigma^\alpha |U|_{2,B(\sigma)} \\ &\quad + c\sigma^{(\alpha+n)/2}\left(\|U\|_{C^{0,\alpha/2}(\overline{B(r)})} + \sum_j \|D_j U\|_{C^{0,\alpha/2}(\overline{B(r)})}\right) \\ &\leqslant c\sigma^{(\alpha+n)/2} K_1. \end{aligned}$$

Therefore inequality (7.7) gives

$$|w|^2_{0,B(\sigma)} \leqslant cK_1^2 |\lambda|^{-2} \sigma^{\alpha+n} \tag{7.10}$$

where K_1 is defined in (7.9) and c is independent of λ and σ.

Next, we remark that the difference $v = U - w \in H^2(B(\sigma), \mathbb{C}^N)$ is a solution of the system λv-$\sum_{ij} A_{ij}(x_0)\, D_{ij} v = \xi$ in $B(\sigma)$, where ξ is the following constant vector of C^N:

$$\xi = g(x_0) - \sum_j B_j(x_0)\, D_j U(x_0) - C(x_0)\, U(x_0).$$

Hence, inequality (4.3) yields, for each $t \in]0, 1]$,

$$|v - v_{B(t\sigma)}|^2_{0,B(t\sigma)} \leqslant ct^{n+2} |v - v_{B(\sigma)}|^2_{0,B(\sigma)} \tag{7.11}$$

where c is independent of λ and σ.

Now, as $U = v + w$ in $B(\sigma)$, from (7.10) and (7.11) we have

$$\begin{aligned} |U - U_{B(t\sigma)}|^2_{0,B(t\sigma)} &\leqslant c|v - v_{B(t\sigma)}|^2_{0,B(t\sigma)} + c|w|^2_{0,B(t\sigma)} \\ &\leqslant ct^{n+2}|v - v_{B(\sigma)}|^2_{0,B(\sigma)} + c|w|^2_{0,B(\sigma)} \\ &\leqslant ct^{n+2}|U - U_{B(\sigma)}|^2_{0,B(\sigma)} + cK_1^2 |\lambda|^{-2}\sigma^{\alpha+n} \end{aligned}$$

for each $t \in]0, 1]$.

Then, in order to get inequality (7.5) one has only to use Lemma 2.13. Inequality (7.6) can be proved by the same procedure. In fact, setting

$$K_2 = [g]_{C^{0,\alpha}(\overline{B(r)})} + \sum_{j=0}^{2} |U|_{j,\infty,B(r)}$$

we get

$$|G|_{0,B(\sigma)} \leqslant c\sigma^{\alpha + n/2} K_2$$

and so

$$w|w|^2_{0,B(\sigma)} \leqslant c K_2^2 |\lambda|^{-2} \sigma^{n+2\alpha}.$$

Hence, taking into account (7.11) and arguing exactly as above we complete the proof of the lemma. ∎

We now need boundary estimates. So, recalling that $\partial\Omega$ is of class $C^{2,\alpha}$, we may confine ourselves to analyzing solutions of the system $(\lambda + E)\, U = g$ in $B^+(1)$. Let us set

$$\lambda_0 = \nu^{-1} \sup_{x \in \overline{B^+(1)}} \sum_i |B_i(x)|^2$$

(see also (2.8)).

LEMMA 7.4. *Let conditions* (7.1) *and* (2.7) *hold in* $B^+(1)$ *and let* $\operatorname{Re} \lambda > \lambda_0$. *Let g be Hölder continuous on $\overline{B^+(1)}$ with exponent α and vanish on $\Gamma(1)$.*

Then, for each solution $U \in H^2_{(n-\alpha)}(B^+(1), \mathbb{C}^N)$ *of the problem*

$$\begin{aligned} (\lambda + E)\, U &= g \quad &&\text{in} \quad B^+(1) \\ U &= 0 \quad &&\text{on} \quad \Gamma(1) \end{aligned}$$

and for each $\rho \in\,]0, 1]$,

$$|U|^2_{0,B^+(\rho)} \leqslant c \left\{ \frac{1}{(|\lambda| - \lambda_0)^2} \left([g]^2_{C^{0,\alpha}(\overline{B^+(1)})} + \sum_{|\beta| \leqslant 2} \|D^\beta U\|^2_{L^{2,n-\alpha}(B^+(1))} \right) + |U|^2_{0,B^+(1)} \right\} \rho^{n+\alpha} \tag{7.12}$$

where c is independent of λ.

Moreover, if $U \in H^{2,\infty}(B^+(1), \mathbb{C}^N)$, *then for each* $\rho \in\,]0, 1]$

$$|U|^2_{0,B^+(\rho)} \leqslant c \left\{ \frac{1}{(|\lambda| - \lambda_0)^2} \left([g]^2_{C^{0,\alpha}(\overline{B^+(1)})} + \sum_{j=0}^{2} |U|^2_{j,\infty,B^+(1)} \right) + |U|^2_{0,B^+(1)} \right\} \rho^{n+2\alpha} \tag{7.13}$$

where c is independent of λ.

Proof. This proof has much in common with that of the previous lemma and so we will only sketch it.

If $0<\sigma\leqslant 1$, then we can solve the problem

$$w\in H^2(B^+(\sigma),\mathbb{C}^N),\qquad w=0 \text{ on } \Gamma(\sigma)$$
$$\lambda w-\sum_{ij}A_{ij}(0)\,D_{ij}w+\sum_j B_j(0)\,D_jw=G,$$

where

$$G=g+\sum_{ij}(A_{ij}-A_{ij}(0))\,D_{ij}U-\sum_j(B_j-B_j(0))\,D_jU-CU,$$

in such a way that

$$(|\lambda|-\lambda_0)|w|_{0,B^+(\sigma)}\leqslant c|G|_{0,B^+(\sigma)}$$

with c independent of λ and σ.

Furthermore

$$|G|_{0,B^+(\sigma)}\leqslant c\left([g]_{C^{0,\alpha}(\overline{B^+(1)})}+\sum_{|\beta|\leqslant 2}\|D^\beta U\|_{L^{2,n-\alpha}(B^+(1))}\right)\sigma^{(n+\alpha)/2}$$

and so

$$|w|^2_{0,B^+(\sigma)}\leqslant\frac{C}{(|\lambda|-\lambda_0)^2}\left([g]^2_{C^{0,\alpha}(\overline{B^+(1)})}+\sum_{|\beta|\leqslant 2}\|D^\beta U\|^2_{L^{2,n-\alpha}(B^+(1))}\right)\sigma^{(n+\alpha)/2}.$$

On the other hand, denoting by v the difference $U-w$,

$$v\in H^2(B^+(\sigma),\mathbb{C}^N),\qquad v=0\quad\text{on } \Gamma(\sigma)$$
$$\lambda v-\sum_{ij}A_{ij}(0)\,D_{ij}v+\sum_j B_j(0)\,D_jv=0.$$

Now, if σ is sufficiently small, i.e., $\sigma\leqslant\sigma_\alpha$, then Corollary 4.8 implies

$$|v|^2_{0,B^+(t\sigma)}\leqslant c\,t^{n+2\alpha}|v|^2_{0,B^+(\sigma)}$$

for each $t\in\,]0,1]$.

Hence. reasoning as in the proof of Lemma 7.3, we obtain inequality (7.12), initially for $\rho\leqslant\sigma_\alpha$ and then in general, as it is trivial when $\rho>\sigma_\alpha$.

Inequality (7.13) may be proved by the same procedure. ▌

Remark 7.5. Combining inequalities (7.5) and (7.12) we get, recalling Theorem 2.1,

$$[U]_{C^{0,\alpha/2}(\overline{B^+(r)})}\leqslant c\left\{(|\lambda|-\lambda_0)^{-1}\left([g]_{C^{0,\alpha}(\overline{B^+(1)})}\right.\right.$$
$$\left.\left.+\sum_{|\beta|\leqslant 2}\|D^\beta U\|_{L^{2,n-\alpha}(B^+(1))}\right)+|U|_{0,B^+(1)}\right\}\qquad(7.14)$$

where $r\in\,]0,1]$ and c is independent of λ.

Similarly, inequalities (7.6) and (7.13) yield

$$[U]_{C^{0,\alpha}(\overline{B^+(r)})} \leqslant c\left\{(|\lambda| - \lambda_0)^{-1}\left([g]_{C^{0,\alpha}(\overline{B^+(1)})} + \sum_{j=0}^{2} |U|_{j,\infty,B^+(1)}\right) + |U|_{0,B^+(1)}\right\}. \tag{7.15}$$

It is now easy to conclude the

Proof of Theorem 7.1. To begin with, Theorem 5.1 implies that, if $\operatorname{Re}\lambda > \omega_0$, then $u \in H^2_{(n-\alpha)}(\Omega, \mathbb{C}^N)$ and

$$\sum_{|\beta| \leqslant 2} \|D^\beta u\|_{L^{2,n-\alpha}(\Omega)} \leqslant c\|f\|_{C^{0,\alpha}(\bar\Omega)} \tag{7.16}$$

where c is independent of λ. Indeed, inequality (7.16) follows from (5.13), (5.14), and (5.15), recalling (2.3) as well.

We then apply Lemma 7.3; by inequality (7.5) and Theorem 2.1

$$\|u\|_{C^{0,\alpha/2}(\bar\Omega)} \leqslant c|\lambda|^{-1}\|f\|_{C^{0,\alpha}(\bar\Omega)} + c|u|_{0,\Omega} \tag{7.17}$$

where $\Omega_0 \subset\subset \Omega$ and c is independent of λ.

To estimate the Hölder norm of u near the boundary of Ω we use a standard argument, already mentioned in Section 5. We consider a finite open cover $\{U_1,..., U_m\}$ of $\partial\Omega$ and a corresponding set $\{\Phi_1,..., \Phi_m\}$ of one-to-one transformations taking U_k onto $B(1)$ for $k = 1,..., m$. Moreover, Φ_k and Φ_k^{-1} have Hölder continuous second-order derivatives with exponent α and satisfy condition (2.2). Therefore Lemma 7.4 and Remark 7.5 may be applied to the transformed system

$$(\lambda + E^k)\, U^k = f^k \qquad \text{on} \quad B^+(1)$$

where $U^k(x) = u(\Phi_k^{-1}(x))$, $x \in B^+(1)$. If $\operatorname{Re}\lambda$ is greater than a certain positive number λ_k, then by inequalities (7.14) and (7.16)

$$\|u\|_{C^{0,\alpha/2}(\Phi_k^{-1}(\overline{B^+(r)}))} \leqslant c\{(|\lambda| - \lambda_k)^{-1}\|f\|_{C^{0,\alpha}(\bar\Omega)} + |u|_{0,\Omega}\} \tag{7.18}$$

where $r \in]0, 1]$ and c is independent of λ.

Let us fix r so that $\{\Phi_1^{-1}(B(r)),..., \Phi_m^{-1}(B(r))\}$ still covers $\partial\Omega$. Combining inequalities (7.17), (7.18), and (3.4) we conclude that, if

$$\operatorname{Re}\lambda > \omega = 2\max\{\omega_0, \lambda_1,..., \lambda_m\}, \tag{7.19}$$

then

$$(|\lambda| - \omega/2)\|u\|_{C^{0,\alpha/2}(\bar\Omega)} \leqslant c_1\|f\|_{C^{0,\alpha}(\bar\Omega)} \tag{7.20}$$

where c_1 is independent of λ.

Inequality (7.20) is not yet our thesis, but could be regarded as a first step towards it. In fact, we can now make use of the well-known $\mathscr{L}^{2,\mu}$ estimates for solutions of elliptic systems with smooth coefficients (see [9, 11]; see also [19, Chap. III]). Since problem (7.2) is equivalent to

$$u \in H^2 \cap H_0^1(\Omega, \mathbb{C}^N)$$
$$(\omega/2 + E)\,u = f(\omega/2 - \lambda)\,u$$

then

$$\sum_{j=0}^{2} |u|_{j,\infty,\Omega} \leqslant \sum_{|\beta| \leqslant 2} \|D^\beta u\|_{C^{0,\alpha/2}(\bar{\Omega})}$$
$$\leqslant c_2\{\|f\|_{C^{0,\alpha}(\bar{\Omega})} + (\omega/2 + |\lambda|)\|u\|_{C^{0,\alpha/2}(\bar{\Omega})}\}$$

where c_2 is independent of λ. So, by (7.19) and (7.20)

$$\sum_{j=0}^{2} |u|_{j,\infty,\Omega} \leqslant c_2(1 + 2c_1)\|f\|_{C^{0,\alpha}(\bar{\Omega})}. \tag{7.21}$$

The final step will be made by gaining the right exponent in (7.20), i.e., showing

$$(|\lambda| - \omega/2)\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant c\|f\|_{C^{0,\alpha}(\bar{\Omega})} \tag{7.22}$$

which in turn implies

$$\sum_{|\beta| = 2} \|D^\beta u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant c\, \|f\|_{C^{0,\alpha}(\bar{\Omega})}.$$

Inequality (7.3) will then follow recalling (2.4).

On the other hand, (7.22) can be obtained by just repeating the procedure that led to (7.20) and using (7.21) instead of (7.16). Estimates (7.6) and (7.18) were proved for this purpose. ∎

Remark 7.6. It would now be easy to prove inequality (7.3) even if ω is replaced by ω_0.

Appendix

In this Appendix we just wish to sketch the proof of inequality (2.12). Our attention will be particularly concentrated on the estimation of the constants c_1 and c_2.

Let us consider the sesquilinear form

$$a_\Omega(u, \phi) = \int_\Omega \left\{ \sum_{ij} (A_{ij} D_j u \,|\, D_i \phi)_N + \sum_j (B_j D_j u \,|\, \phi)_N + (Cu \,|\, \phi)_N \right\} dx.$$

We briefly recall how differentiability is obtained in two particular cases.

LEMMA 1. *Let conditions* (2.6) *and* (2.7) *hold on a ball* $B(R)$ *and let* A_{ij} *be of class* C^1 *in this ball. Let* $f \in L^2(B(R), \mathbb{C}^N)$ *and* $\operatorname{Re} \lambda \geqslant 0$. *Then each solution of the problem*

$$\begin{aligned} &u \in H_0^1(B(R), \mathbb{C}^N) \\ &\lambda(u, \phi)_{0,B(R)} + a_{B(R)}(u, \phi) = (f, \phi)_{0,B(R)}, \qquad \forall \phi \in H_0^1(B(R), \mathbb{C}^N) \end{aligned} \tag{1}$$

belongs to the space $H^2(B(r), \mathbb{C}^N)$ *for each* $r \in\,]0, R[$ *and*

$$|u|_{2,B(r)} \leqslant k_1 \|u\|_{1,B(R)} + (2/\nu)|f|_{0,B(R)} \tag{2}$$

where k_1 *is independent of* λ *and* ν *is the constant that appears in* (2.7).

Proof. We use a standard translation technique.
Let $\theta \in C_0^\infty(B(R))$ be such that

$$0 \leqslant \theta \leqslant 1, \theta \equiv 1 \text{ in } B(r), \theta \equiv 0 \text{ out of } B((r+R)/2).$$

Let us set, for $|t| < (R-r)/4$ and $s = 1,..., n$,

$$\tau_{s,t} u(x) = \frac{u(x + te^s) - u(x)}{t}.$$

Directly from system (1) we have, for each $\phi \in H_0^1(B(R), C^N)$,

$$\begin{aligned} &\operatorname{Re}\{\lambda(\tau_{s,t}(\theta u), \phi)_{0,B(R)} + a_{B(R)}(\tau_{s,t}(\theta u), \phi)\} \\ &\quad \leqslant |a_{B(R)}(\tau_{s,t}(\theta u), \phi) - a_{B(R)}(\theta u, \tau_{s,-t}\phi)| \\ &\qquad + |a_{B(R)}(\theta u, \tau_{s,-t}\phi) - a_{B(R)}(u, \theta\tau_{s,-t}\phi)| + |(f, \theta\tau_{s,-t}\phi)_{0,B(R)}|. \end{aligned}$$

By standard calculations (see, for instance, [11, Chap. II, No. 21]) the previous inequality yields

$$\begin{aligned} &\operatorname{Re}\{\lambda(\tau_{s,t}(\theta u), \phi)_{0,B(R)} + a_{B(R)}(\tau_{s,t}(\theta u), \phi)\} \\ &\quad \leqslant (c\|u\|_{1,B(R)} + |f|_{0,B(R)})|\phi|_{1,B(R)}. \end{aligned}$$

Now, choosing $\phi = \tau_{s,t}(\theta u)$ and recalling (2.9) we get, as $\operatorname{Re} \lambda \geqslant 0$,

$$\begin{aligned} (\nu/2)|\tau_{s,t}(\theta u)|_{1,B(R)}^2 \leqslant{}& (c\|u\|_{1,B(R)} + |f|_{0,B(R)})|\tau_{s,t}(\theta u)|_{1,B(R)} \\ &+ (\lambda_0 + \Lambda)|\tau_{s,t}(\theta u)|_{0,B(R)}^2 \end{aligned}$$

whence inequality (2) easily follows. ∎

LEMMA 2. *Let conditions* (2.6) *and* (2.7) *hold on* $B^+(1)$ *and let* A_{ij} *be of class* C^1 *in this half ball. Let* $f \in L^2(B^+(1), \mathbb{C}^N)$ *and* $\operatorname{Re} \lambda \geqslant 0$. *If* $u \in H^1(B^+(1), \mathbb{C}^N)$ *is a solution of the system*

$$\lambda(u, \phi)_{0, B^+(1)} + a_{B^+(1)}(u, \phi) = (f, \phi)_{0, B^+(1)}, \qquad \forall \phi \in H_0^1(B^+(1), \mathbb{C}^N) \tag{3}$$

and if $u = 0$ *on* $\Gamma(1)$, *then for each* $r \in]0, 1[$

$$D_s u \in H^1(B^+(r), \mathbb{C}^N), \qquad s = 1, \ldots, n-1,$$

and

$$|D_s u|_{1, B^+(r)} \leqslant k_2 \|u\|_{1, B^+(1)} + (2/\nu)|f|_{0, B^+(1)} \tag{4}$$

where k_2 *is independent of* λ *and* ν *is the constant that appears in* (2.7).

The proof of Lemma 2 is completely analogous to that of Lemma 1 and so will be omitted.

Remark. From system (3) we have

$$D_{nn} u = A_{nn}^{-1} \left\{ [\lambda + C]\, u + \sum_j \left[B_j - \sum_i D_i A_{ij} \right] D_j u - \sum_{ij}^{*} A_{ij} D_{ij} u - f \right\}$$

where $\sum_{ij}^{*}$ means summation over all indexes except $ij = nn$.

So $D_{nn} u \in L^2(B^+(r), \mathbb{C}^N)$ and

$$|D_{nn} u|_{0, B^+(r)} \leqslant \frac{1}{\nu} \left(|\lambda|\, |u|_{0, B^+(r)} + |f|_{0, B^+(r)} \right) + k_3 \|u\|_{1, B^+(r)} + \frac{1}{\nu} L^* \left(\sum_{j=1}^{n-1} |D_j u|_{1, B^+(r)}^2 \right)^{1/2} \tag{5}$$

where k_3 is independent of λ and

$$L^* = \sup_{x \in B^+(1)} \left(\sum_{ij} |A_{ij}(x)|^2 \right)^{1/2}.$$

Finally, from (4) and (5) we get

$$|u|_{2, B^+(r)} \leqslant k_4 \|u\|_{1, B^+(1)} + k_5 |f|_{0, B^+(1)} + (|\lambda|/\nu) |u|_{0, B^+(1)} \tag{6}$$

where k_4 is independent of λ and $k_5 = k_5(L^* \vee \nu^{-1})$.

Let now u be a solution of problem (2.10) under the hypotheses of Theorem 2.7. Assume $\operatorname{Re} \lambda \geqslant 0$ as well.

Then by a standard covering argument, the two lemmas and the remark above yield

$$u \in H^2(\Omega, \mathbb{C}^N).$$

From (2) and (6) we also get, by a well-known interpolation inequality,

$$|u|_{2,\Omega} \leqslant (k_6 + k_7|\lambda|)|u|_{0,\Omega} + k_8|f|_{0,\Omega}$$

where k_6 is independent of λ, $k_7 = k_7(\Omega)\, \nu^{-1}$, and $k_8 = k_8(\Omega, L \vee \nu^{-1})$.

Moreover, if $\operatorname{Re} \lambda \geqslant \lambda_0 + \Lambda$, then by Remark 2.6

$$|u|_{2,\Omega} \leqslant (k_6 + k_7\lambda_0)|u|_{0,\Omega} + (k_8 + k_7 c)|f|_{0,\Omega}$$

where $c = c(L \vee \nu^{-1})$.

Theorem 2.7 is now completely proved.

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